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Teoría de las ambigüedades para resoluciones proyectivas de álgebras asociativas.

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Teoría de las ambigüedades para resoluciones proyectivas de álgebras asociativas

En esta tesis estudiamos el problema de calcular resoluciones proyectivas de álgebras asociativas. Nuestro punto de partida es la resolución de Bardzell para álgebras monomiales. Dada un álgebra asociativa, utilizamos el principio de sistemas de reducción de Bergman para asociarle álgebras monomiales. Mostramos que los diferenciales de la resolución de Bardzell de estas álgebras pueden modificarse para obtener resoluciones proyectivas del álgebra de partida. Mas aún, damos un criterio para que un complejo proveniente de una modificación de la resolución de Bardzell de un álgebra monomial asociada sea exacto. Aplicamos nuestro método a tres familias de álgebras: las intersecciones completas cuánticas, las álgebras de Weyl generalizadas cuánticas y las álgebras down-up. En el caso de las álgebras down-up, utilizamos la resolución obtenida para calcular invariantes homológicos de estas álgebras. De esta manera probamos propiedades de regularidad y damos una solución al problema de isomorfismo para las álgebras down-up no noetherianas.

Palabras clave: álgebras asociativas, cohomología de Hochschild, resoluciones proyectivas.

Theory of ambiguities for projective resolutions of associative algebras

This thesis is concerned with the problem of computing projective resolutions of associative algebras. Our starting point is Bardzell's resolution for monomial algebras. Given an associatve algebra, we use Bergman's principle of reduction systems to associate monomial algebras to it. We prove that the differentials in Bardzell's resolution of these monomial algebras can be modified to obtain projective resolutions of the original algebra. We also give sufficient conditions for a complex coming from a modification of Bardzell's resolution of an associated monomial algebra to be exact. We apply our method to three families of algebras: Quantum complete intersections, Quantum generalized Weyl algebras and down-up algebras. In the case of down-up algebras, we use the resolution obtained to compute homological invariants of these algebras. This way we prove regularity properties and we solve the isomorphism problem for non-noetherian down-up algebras.

Keywords: associative algebras, Hochschild cohomology, projective resolutions.

Contents

Introduction

The study of invariants and intrinsic properties of a given object is central in every field in mathematics. Description of these kind of properties are often used to answer elementary but difficult questions, such as deciding whether two given objects are equivalent or not. In the theory of associative algebras these invariants usually come in the form of homology groups. Among others, Hochschild homology and cohomology groups are a very powerful tool and have been widely studied during the last decades. To compute these invariants for an algebra A one needs a projective bimodule resolution of A as a bimodule over itself. Every associative algebra has a standard resolution, called the *bar resolution*, but it is almost impossible to perform computations using it. Therefore, the first problem one faces when computing Hochschild (co)homology is to find a convenient projective resolution of the given algebra. The results of this thesis belong to this domain. The approach that we take follows work of Anick, Green and Bardzell.

Let k be a field, A a k-algebra presented by generators and relations and T a one-dimensional A-module. In [1] and [2] the authors construct explicit left A-modules from combinatorial objects called *ambiguities*, coming from the way the relations interact, and prove that there exist morphisms between them forming a projective A-module resolution of T. In [4] and [5] Bardzell provided explicit modules and differentials of a projective bimodule resolution for monomial algebras. These are algebras of the form $A = kQ/I$ with Q a quiver, kQ the path algebra over Q and I a two-sided ideal generated by paths of lenght at least two. The modules are also constructed from ambiguities.

We deal with the case when the ideal I is not monomial. In this setting we find that Bergman's language of *reduction systems*, developed in [3], provides a natural framework to reduce the problem to the monomial case. Roughly speaking, given an associative k-algebra $A = kQ/I$, with I an arbitrary two-sided ideal, we construct monomial algebras $A_S = kQ/I_S$ which are strongly related to A and we prove that for each of these monomial algebras it is possible to modify the formulas of the differentials in their Bardzell's resolution in a very controlled way to obtain differentials between A-bimodules, again constructed from ambiguities, which form a projective A-bimodule resolution of A. With this we recover the results in [1] and in [2] with significant improvements on the information obtained about the differentials. Following an observation in [3], we construct partial orders on the set of paths that help us control this process.

The proof of this existence theorem goes by induction, and thus it is very difficult to trace back the construction of the differentials to actually compute these resolutions in examples. To deal with the problem of effective computations we obtain sufficient conditions for morphisms between these ambiguity modules to form a resolution, and the existence theorem says that there always exist resolutions verifying these conditions. This gives a very general method to compute bimodule resolutions of algebras presented by generators and relations. In Chapters 4 and 5 we apply our method in the following families of algebras.

- *Quantum complete intersections*. The members of this family are the algebras $k\langle x,y\rangle/J(\xi,n,m)$, where $\xi \in k$, n and m are integers at least equal to 2 and $J(\xi,n,m)$ is the two-sided ideal generated by the elements x^n, y^m and $yx-\xi xy$. When $n=m=2$, the algebras $k\langle x,y\rangle/J(\xi,n,m)$ are Koszul and with our method we recover their Koszul resolution. For the general case these algebras are no longer Koszul but our method applies with no further difficulties. The formulas we obtain are a natural generalization of the formulas for the case $n=m=2$.
- *Quantum generalized Weyl algebras*. This is the family of algebras defined by $k\langle y,x,h\rangle/J(a(h),q)$, where $a(h)$ is a polynomial in the variable h, $q \in k^\times$ and $J(a,q)$ is the two-sided ideal generated by the elements $hy-qyh$, $hx-q^{-1}xh$, $yx-a(h)$ and $xy-a(qh)$.
- *Down-up algebras*. Given elements α,β and γ in a field k, the *down-up* algebra $A(\alpha,\beta,\gamma)$ is the algebra with generators d and u subject to the relations $d^2u-\alpha dud-\beta ud^2-\gamma d$ and $du^2-\alpha udu-\beta ud^2-\gamma u$. These algebras are 3-Koszul if and only if $\gamma=0$. Our method applies to a general down-up algebra $A(\alpha,\beta,\gamma)$ and we obtain in all cases a resolution of length 3. In the cases where the algebra is 3-Koszul our resolution coincides with the Koszul resolution.

In the cases of Quantum complete intersections and Quantum generalized Weyl algebras, the resolutions we obtain were previously constructed in [8] and [33]. In both cases the authors use very specific methods. With these three families we aim to show the flexibility of our method and the generality in which it can be applied.

The techniques described in this thesis may also be used to compute cup products and Gerstenhaber brackets on Hochschild cohomology, for which a more refined study of the contracting homotopies is needed. For researchers interested in Gerstenhaber deformations of associative algebras, Proposition 3.22 is of particular importance since it provides strong information to compute Hochschild cohomology in degree 2.

The article [18] contains part of the results of this thesis.

Resumen de la tesis

A continuación resumimos los puntos principales de la tesis. Parte de estos resultados dieron origen al artículo [18].

Capítulo 1: sistemas de reducción, órdenes parciales y ambigüedades

En este capítulo definimos y estudiamos las herramientas que utilizaremos en los siguientes capítulos para construir resoluciones proyectivas de álgebras asociativas. Se trata de *sistemas de reducción* y *ambigüedades*.

Sistemas de reducción

Comenzamos por establecer la notación. Sean k un cuerpo y Q un carcaj con un conjunto finito de vértices. Denotamos $k^{\times} = k \setminus \{0\}$. Dado $n \in \mathbb{N}$, Q_n denota el conjunto de caminos de Q de largo n y $Q_{\leq n}$ el conjunto de caminos de largo a lo sumo n. Si $c \in Q_n$, escribimos $|c| = n$. Si $a, b, p, q \in Q_{\geq 0}$ son tales que $q = apb$, decimos que p es un divisor de q. Si $a \in Q_0$ decimos que p es un *divisor a izquierda de* q, y si $b \in Q_0$ decimos que p es un *divisor a derecha de* q. Denotamos $t, s : Q_1 \longrightarrow Q_0$ las funciones *destino* y *origen* usuales. Dado $s \in Q_{\geq 0}$ y un elemento $f = \sum_i \lambda_i c_i \in kQ$ tal que $c_i \in Q_{\geq 0}$ y $t(s) = t(c_i)$, $s(s) = s(c_i)$ para todo i, decimos que f es *paralelo* a s. Sea $E := kQ_0$ la subálgebra de kQ generada por los vértices de Q.

Dado un anillo R, un R-módulo a izquierda M y un conjunto $X \subseteq M$, denotamos $\langle X \rangle_R$ el R-submódulo a izquierda de M generado por X.

Utilizaremos cierta terminología de [3]. Sea $\mathcal{R}$ un subconjunto de $Q_{\geq 0} \times kQ$. Decimos que $\mathcal{R}$ es *un sistema de reducción* si para todo $(s, f) \in \mathcal{R}$, el elemento f es paralelo a s y $s \neq f$. Una terna (a, ρ, c), donde $\rho = (s, f) \in \mathcal{R}$ y $a, c \in Q_{\geq 0}$ son tales que $asc \neq 0$ en kQ, se llama *reducción básica* y la denotamos $r_{a,\rho,c}$. Observamos que $r_{a,\rho,c}$ determina un endomorfismo $r_{a,\rho,c} : kQ \longrightarrow kQ$ dado por $r_{a,\rho,c}(asc) = afc$ y $r_{a,\rho,c}(q) = q$ para todo $q \neq asc$. En caso de que no se preste a confusión escribiremos $r_{a,s,c}$ en lugar de $r_{a,\rho,c}$. Una *reducción* es una n-upla $(r_n, \dots, r_1)$, donde $n \in \mathbb{N}$ y r_i es una reducción básica para todo i. Toda reducción determina un endomorfismo $r : kQ \longrightarrow kQ$ dado por la composición $r_n \circ \cdots \circ r_1$ de los endomorfismos correspondientes a las reducciones básicas $r_n, \ldots, r_1$.

Un elemento $x \in kQ$ se dice *irreducible* para $\mathcal{R}$ si $r(x) = x$ para toda reducción básica r. Un camino $p \in Q_{\geq 0}$ se dice de *reducción finita* si para cada sucesión infinita de reducciones básicas $(r_i)_{i \in \mathbb{N}}$ existe $n_0 \in \mathbb{N}$ tal que $r_n \circ \cdots \circ r_1(p) = r_{n_0} \circ \cdots \circ r_1(p)$ para todo $n \geq n_0$. Un camino p se dice de *reducción única* si es de reducción finita y además para cada par de reducciones r y r' tales que $r(p)$ y $r'(p)$ son elementos irreducibles, vale la igualdad $r(p) = r'(p)$.

Un orden parcial

Sean k un cuerpo y Q un carcaj como antes. Sea $\mathcal{R}$ un sistema de reducción tal que todo camino es de reducción finita.

Sea $x = \sum_{i=1}^{n} \lambda_i c_i \in kQ$ con $\lambda_1, \ldots, \lambda_n \in k^\times$ y $c_1, \ldots, c_n$ caminos de largo mayor o igual que 0. El conjunto $\{c_1, \ldots, c_n\}$ se llama el *soporte de* x y lo denotamos $\mathsf{Su}(x)$.

Definimos una relación $\preceq$ en el conjunto $k^\times Q_{\geq 0} := \{\lambda p \in kQ : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$ como la menor relación reflexiva y transitiva que cumple $\lambda p \preceq \mu q$ si existe una reducción r tal que $r(\mu q) = \lambda p + x$ con $p \notin \mathsf{Su}(x)$. Definimos $0 \preceq \lambda p$ para todo $\lambda p \in k^\times Q_{\geq 0}$. Sean $x \in kQ$ y $\lambda p \in k^\times Q_{\geq 0}$. Si $x = \sum_{i=1}^{n} \lambda_i c_i$ con $\lambda_i \in k^\times$ para todo i y $\lambda_i p_i \preceq \lambda p$ para todo i, escribimos $x \preceq \lambda p$. Si además $x \neq \lambda_i p_i$ para todo i, escribimos $x \prec p$.

Lema. *La relación binaria $\preceq$ es un orden parcial que satisface la condición de cadena descendente.*

La condición Diamante

Sea I un ideal bilátero de kQ. Denotamos con π a la proyección canónica $\pi : kQ \longrightarrow kQ/I$. Sea $\mathcal{R}$ un sistema de reducción. Decimos que $\mathcal{R}$ cumple la *condición Diamante para* I si

1. el ideal I es igual al ideal bilátero generado por el conjunto $\{s - f\}_{(s,f) \in \mathcal{R}}$,
2. todo camino es de reducción única y
3. para cada $(s, f) \in \mathcal{R}$, el elemento f es irreducible.

Una consecuencia del Lema del diamante de Bergman es el siguiente lema.

Lema. *Si $\mathcal{R}$ es un sistema de reducción que satisface la condición Diamante para I, entonces el conjunto $\mathcal{B}$ de caminos irreducibles cumple las siguientes propiedades.*

- *Si* $b \in \mathcal{B}$ *y* $q \in Q_{\geq 0}$ *es un elemento que divide a* b*, entonces* $q \in \mathcal{B}$.

- $\pi(b) \neq \pi(b')$ *para todos* $b, b' \in \mathcal{B}$ *tal que* $b \neq b'$.

- $\{\pi(b) : b \in \mathcal{B}\}$ *es una* k*-base de* kQ/I.

Este lema es una de las principales razones por las cuales los sistemas de reducción que cumplen la condición Diamante son tan útiles para nuestros propósitos. La siguiente proposición garantiza que todo ideal bilátero de kQ posee sistemas de reducción de este tipo.

Proposición. *Sea* I *un ideal bilátero de* kQ*. Existe un sistema de reducción* $\mathcal{R}$ *que cumple la condición Diamante para* I.

Sea $\mathcal{R}$ un sistema de reducción. A continuación daremos algunas definiciones más.

- Una *ambigüedad de inclusión* es una 5-upla $(\rho_1, \rho_2, a, b, c)$ con $\rho_1, \rho_2 \in \mathcal{R}$, $a, b, c \in Q_{\geq 0}$, tales que $\rho_1 = (abc, f_1)$ y $\rho_2 = (b, f_2)$ para ciertos $f_1, f_2 \in kQ$.

- Denotamos con $S_{\mathcal{R}}$ al conjunto $\{s \in Q_{\geq 0} :$ existe $f \in kQ$ tal que $(s, f) \in \mathcal{R}\}$.

El siguiente resultado dice que todo sistema de reducción que cumple la condición Diamante para un ideal I se puede modificar para que no tenga ambigüedades de inclusión.

Proposición. *Sean* I *un ideal bilátero de* kQ *y* $\mathcal{R}$ *un sistema de reducción que cumple la condición Diamante para* I*. El conjunto* $\mathcal{R}' := \{(s, f) \in \mathcal{R} :$ *no existe una ambigüedad de inclusión* $(\rho_1, \rho_2, a, b, c)$ *tal que* $abc = s\}$ *es un sistema de reducción que cumple la condición Diamante para* I *y no tiene ambigüedades de inclusión.*

Mas aún, si $\mathcal{R}$ es un sistema de reducción sin ambigüedades de inclusión que cumple la condición Diamante para un ideal I, entonces se puede modificar para obtener un sistema de reducción $\mathcal{R}'$ que cumpla la condición Diamante, no tenga ambigüedades de inclusión y que verifique $S_{\mathcal{R}'} \subseteq Q_{\geq 2}$.

Ambigüedades

Sean I un ideal bilátero de kQ y $\mathcal{R}$ un sistema de reducción sin ambigüedades de inclusión, que cumple la condición Diamante y tal que $S_{\mathcal{R}} \subseteq Q_{\geq 2}$. Recordamos la definición de n-ambigüedad, que se puede encontrar en [1], [2], [4] y [32].

Definición. Dados $n \geq 2$ y $p \in Q_{\geq 0}$,

1. el camino p es una n-*ambigüedad a izquierda* si existen $u_0 \in Q_1$ y caminos irreducibles $u_1, \dots, u_n$ tales que

(a) $p = u_0 u_1 \cdots u_n$,

(b) para todo i, el camino $u_i u_{i+1}$ no es irreducible pero $u_i d$ es irreducible para todo divisor a izquierda d de u_{i+1}, distinto de u_{i+1}.

2. El camino p es una n-ambigüedad a derecha si existe $v_0 \in Q_1$ y caminos irreducibles $v_1, \ldots, v_n$ tales que

   (a) $p = v_n \cdots v_0$,

   (b) para todo i, el camino $v_{i+1} v_i$ no es irreducible pero $d v_i$ es irreducible para todo divisor a derecha d de v_{i+1}, distinto de v_{i+1}.

Definimos $\mathcal{A}_{-1} := Q_0$, $\mathcal{A}_0 := Q_1, \mathcal{A}_1 := S_{\mathcal{R}}$ y para todo $n \geq 2$, definimos $\mathcal{A}_n$ y $\mathcal{A}'_n$ los conjuntos de n-ambigüedades a izquierda y a derecha, respectivamente.

Proposición. *Para todo $n \geq 2$ vale la igualdad $\mathcal{A}_n = \mathcal{A}'_n$. Además, $\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ si n y m son distintos.*

Capítulo 2: trabajos previos de Anick, Green y Bardzell

Este capítulo contiene un resumen de los trabajos previos de Anick, Green y Bardzell sobre la construcción de resoluciones proyectivas. Ver [1], [2], [4] y [32].

La resolución de Anick

Sea A una k-álgebra presentada de la forma $A = k\langle X \rangle / I$, donde X es un conjunto de generadores de A e I es un ideal de $k\langle X \rangle$. Sean $\leq$ un orden total en X y $\omega : X \longrightarrow \mathbb{N}$ una función. Estos datos inducen un orden total en el conjunto de monomios en X, llamado *deglex*, que denotamos $\leq_\omega$. Se trata de un orden lexicográfico con pesos. Anick probó que existen conjuntos de monomios $\mathcal{A}_n$, con $n \geq 2$, tales que para todo A-módulo simple T, existe una resolución de T por A-módulos a derecha libres de la forma:

$$0 \longleftarrow T \xleftarrow{\epsilon} A \xleftarrow{d_0} kX \otimes_k A \xleftarrow{d_1} kS \otimes_k A \xleftarrow{d_2} k\mathcal{A}_2 \otimes_k A \xleftarrow{d_3} \cdots,$$

donde S es un conjunto minimal de generadores de I y kY denota el k-espacio vectorial generado por un conjunto de caminos Y. Con nuestra notación, $\mathcal{A}_n$ corresponde a las n-ambigüedades determinadas por un sistema de reducción $\mathcal{R}$ que se construye a partir de $\leq_\omega$. La única información sobre los diferenciales d_i es la siguiente. El orden en los monomios $\leq_\omega$ se puede extender a un orden total en los conjuntos de tensores elementales de $k\mathcal{A}_n \otimes_k A$ de manera tal que para todo n, todos los términos de $d_n(u_0 \cdots u_n \otimes 1) - u_0 \cdots u_{n-1} \otimes u_n$ son estrictamente menores que $u_0 \cdots u_{n-1} \otimes u_n$, para todo $p = u_0 \cdots u_n \in \mathcal{A}_n$.

La resolución de Anick-Green es una generalización de este resultado al contexto de álgebras de caminos.

La resolución de Bardzell

Sea Q un carcaj con una cantidad finita de vértices. Un ideal bilátero de kQ se dice monomial si está generado por caminos de largo por lo menos igual a 2. Una k-álgebra se dice monomial si existe un carcaj Q y un ideal monomial I de kQ tales que $A \cong kQ/I$.

Sea $A = kQ/I$ con I ideal monomial. Denotemos con $\pi : kQ \longrightarrow A$ la proyección canónica. Sea S el conjunto de caminos $s \in I$ tales que $s' \notin I$ para todo divisor s' de s, con $s' \neq s$. El ideal I está generado por el conjunto S. Bardzell probó que existen conjuntos de caminos $\mathcal{A}_n$ para todo $n \geq 2$ que dan lugar a una resolución de A por A-bimódulos proyectivos, dando fórmulas explícitas para los diferenciales. Con nuestra notación, los conjuntos $\mathcal{A}_n$ corresponden a las n-ambigüedades del sistema de reducción $\mathcal{R} = \{(s, 0) : s \in S\}$. La resolución de Bardzell es la siguiente

$$\begin{array}{ccccccc} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A & \xrightarrow{d_0} & A \otimes_E A & \xrightarrow{d_{-1}} & A & \longrightarrow & 0 \\ & & \downarrow \cong & & & & \\ & & A \otimes_E k\mathcal{A}_{-1} \otimes_E A & & & & \end{array}$$

donde $\mathcal{A}_{-1} = Q_0$, $\mathcal{A}_0 = Q_1$, $\mathcal{A}_1 = S$, y

1. $d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ para $n \geq 0$,
2. $d_{-1}(a \otimes b) = ab$ es la multiplicación en A,
3. si n es par, $q \in \mathcal{A}_n$ y $q = u_0 \cdots u_n = v_n \cdots v_0$ son respectivamente las factorizaciones de q como n-ambigüedad a izquierda y a derecha,
$$d_n(1 \otimes q \otimes 1) = \pi(v_n) \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes \pi(u_n),$$
4. si n es impar y $q \in \mathcal{A}_n$,
$$d_n(1 \otimes q \otimes 1) = \sum_{\substack{apc=q \\ p \in \mathcal{A}_{n-1}, a,c \in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c).$$

Esta resolución tiene una homotopía de contracción $s_\bullet$ dada por las siguientes fórmulas.

Para $n = -1$, $s_{-1} : A \longrightarrow kQ \otimes_E k\mathcal{A}_{-1} \otimes_E A$ es el morfismo de $kQ - E$-bimódulos definido por $s_{-1}(a) = a \otimes 1$, con $a \in kQ$. Para $n \in \mathbb{N}_0$, $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ es

$$s_n(1 \otimes q \otimes \pi(b)) = (-1)^{n+1} \sum_{\substack{apc=qb \\ p\in\mathcal{A}_n, a,c\in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c),$$

con $b \in \mathcal{B}$ y $q \in \mathcal{A}_{n-1}$. Esta homotopía de contracción fue encontrada por Sköldberg en [32].

Capítulo 3: resoluciones proyectivas usando ambigüedades

Este es el capítulo central de la tesis. En él enunciamos y demostramos nuestros principales teoremas.

Sean Q un carcaj e I un ideal bilátero de kQ. Llamamos $A = kQ/I$ y $\pi : kQ \longrightarrow A$ la proyección canónica. Sea $\mathcal{R}$ un sistema de reducción sin ambigüedades de inclusión que cumple la condición Diamante para I, tal que $S_{\mathcal{R}} \subseteq Q_{\geq 2}$. Denotamos $S = S_{\mathcal{R}}$ y $\mathcal{B} = \{p \in Q_{\geq 0} : p \text{ es irreducible}\}$.

Asociamos a A la k-álgebra monomial $A_S := kQ/\langle S\rangle$. Sea $\pi' : kQ \longrightarrow A_S$ la proyección canónica. El conjunto de n-ambigüedades del sistema de reducción $\mathcal{R}' = \{(s, 0) : s \in S\}$ coincide con las n-ambigüedades de $\mathcal{R}$. El conjunto $\mathcal{B}$ de caminos irreducibles para $\mathcal{R}$ es igual al conjunto de caminos irreducibles para $\mathcal{R}'$. Luego, hay morfismos de k-espacios vectoriales $i : A \longrightarrow kQ$ e $i' : A_S \longrightarrow kQ$ tales que $i(\pi(b)) = b$ e $i'(\pi'(b)) = b$ para todo $b \in \mathcal{B}$. Definimos $\beta = i \circ \pi$ y $\beta' = i' \circ \pi'$. Para cada $n \geq -1$ consideramos los siguientes morfismos k-lineales.

$$\begin{aligned} \pi_n &:= \pi \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi, & \pi'_n &:= \pi' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi', \\ i_n &:= i \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i, & i'_n &:= i' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i', \\ \beta_n &:= i_n \circ \pi_n, & \beta'_n &:= i'_n \circ \pi'_n. \end{aligned}$$

Basándonos en las fórmulas de los diferenciales de Bardzell vamos a definir morfismos de A-bimódulos $\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$. Para esto construimos primero los siguientes morfismos f_n y S_n. Observamos que el kQ-bimódulo $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ es un k-espacio vectorial con base $\{a \otimes q \otimes c : a, c \in Q_{\geq 0}, q \in \mathcal{A}_n, aqc \neq 0 \in kQ\}$. Consideramos el diagrama,

$$\cdots \underset{S_2}{\overset{f_2}{\rightleftarrows}} kQ \otimes_E k\mathcal{A}_1 \otimes_E kQ \underset{S_1}{\overset{f_1}{\rightleftarrows}} kQ \otimes_E k\mathcal{A}_0 \otimes_E A \underset{S_0}{\overset{f_0}{\rightleftarrows}} kQ \otimes_E kQ \underset{S_{-1}}{\overset{f_{-1}}{\rightleftarrows}} kQ \longrightarrow 0$$

$$kQ \otimes_E kQ \xrightarrow{\ \cong\ } kQ \otimes_E k\mathcal{A}_{-1} \otimes_E kQ$$

donde

1. $f_{-1}(a \otimes b) = ab$,

2. si n es par, entonces definimos f_n como el único morfismo k-lineal tal que para todos $a, c \in Q_{\geq 0}$, $q \in \mathcal{A}_n$ tales que $aqc \neq 0$ en kQ y $q = u_0 \cdots u_n = v_n \cdots v_0$ son respectivamente las factorizaciones de q como n-ambigüedad a izquierda y a derecha,

$$f_n(a \otimes q \otimes c) = av_n \otimes v_{n-1} \cdots v_0 \otimes c - a \otimes u_0 \cdots u_{n-1} \otimes u_n c,$$

   y observamos que este morfismo es de kQ-bimódulos.

3. Si n es impar, entonces f_n es el único morfismo k-lineal tal que para todos $a, c \in Q_{\geq 0}, q \in \mathcal{A}_n$ como antes,

$$f_n(a \otimes q \otimes c) = \sum_{\substack{a'pc'=q \\ p \in \mathcal{A}_{n-1}, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'c,$$

   y resulta un morfismo de kQ-bimódulos.

4. Definimos $S_{-1}(x) = x \otimes 1$ y si $n \geq 0$, S_n es el único morfismo k-lineal tal que para todos $a, c \in Q_{\geq 0}, q \in \mathcal{A}_{n-1}$,

$$S_n(a \otimes q \otimes c) = (-1)^{n+1} \sum_{\substack{a'pc'=qc \\ p \in \mathcal{A}_n, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'.$$

   Observamos que S_n es un morfismo de $kQ - E$-bimódulos para todo $n \geq -1$.

Los morfismos f_n inducen morfismos de A-bimódulos $\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ y A_S-E-bimódulos $\delta'_n : A_S \otimes_E k\mathcal{A}_n \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S$ de la siguiente manera

$$\delta_n := \pi_{n-1} \circ f_n \circ i_n,$$
$$\delta'_n := \pi'_{n-1} \circ f_n \circ i'_n.$$

Para $n = -1$ interpretamos estas fórmulas como $\delta_{-1} := \pi \circ f_{-1} \circ i_{-1}$ y $\delta'_{-1} := \pi' \circ f_{-1} \circ i'_{-1}$. Observamos que estas son las respectivas multiplicaciones de A y A_S.

La siguiente sucesión es la resolución de Bardzell de A_S.

$$\cdots \xrightarrow{\delta'_2} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E A_S \xrightarrow{\delta'_{-1}} A_S \longrightarrow 0 \,,$$

Los morfismos S_n inducen morfismos de $A - E$-bimódulos $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$, y morfismos de $A_S - E$-bimódulos $s'_n : A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_n \otimes_E A_S$ de la siguiente manera

$$s_n := \pi_n \circ S_n \circ i_{n-1},$$
$$s'_n := \pi'_n \circ S_n \circ i'_{n-1}.$$

Para $n = -1$ interpretamos estas definiciones como $s_{-1} := \pi_{-1} \circ S_{-1} \circ i$ y $s'_{-1} := \pi'_{-1} \circ S_{-1} \circ i'$. Los morfismos s'_n son la homotopía de contracción de Sköldberg para la resolución de Bardzell.

Recordamos que $k^\times Q_{\geq 0} := \{\lambda p \in kQ : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$. Para cada $n \geq -1$ y $\mu q \in k^\times Q_{\geq 0}$, definimos los siguientes subconjuntos de $A \otimes_E k\mathcal{A}_n \otimes_E A$.

- $\overline{\mathcal{L}}_n^{\preceq}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \preceq \mu q\}$,
- $\overline{\mathcal{L}}_n^{\prec}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \prec \mu q\}$.

Estamos listos para enunciar los teoremas. El primer teorema dice que si tenemos morfismos $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ que forman un complejo y sus fórmulas están dadas por ciertas *deformaciones* de las fórmulas de los morfismos de la resolución de Bardzell de A_S, entonces necesariamente este complejo es exacto. El segundo teorema garantiza que este tipo de resoluciones siempre existe.

Teorema (3.5). *Definimos* $d_{-1} := \delta_{-1}$ *y* $d_0 := \delta_0$. *Dado* $N \in \mathbb{N}_0$ *y morfismos de* A*-bimódulos* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ *para* $1 \leq i \leq N$, *si*

1. $d_{i-1} \circ d_i = 0$ *para todo* i, $1 \leq i \leq N$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}_{i-1}^{\prec}(q) \rangle_k$ *para todo* $i \in \{1, \dots, N\}$ *y para todo* $q \in \mathcal{A}_i$,

entonces el complejo

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

es exacto.

Teorema (3.6). *Existen morfismos de* A*-bimódulos* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ *para* $i \in \mathbb{N}_0$ *y* $d_{-1} : A \otimes_E A \longrightarrow A$ *tales que*

1. $d_{i-1} \circ d_i = 0$, *para todo* $i \in \mathbb{N}_0$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}_{i-1}^{\prec}(q) \rangle_{\mathbb{Z}}$ *para todo* $i \geq -1$ *y* $q \in \mathcal{A}_i$.

Morfismos en grados bajos

Podemos dar una descripción explícita de los diferenciales para estas resoluciones en grados $0, 1, 2$. Al igual que antes, $A = kQ/I$ es una k-álgebra y $\mathcal{R}$ es un sistema de reducción sin ambigüedades de inclusión, que cumple la condición Diamante, tal que $S_{\mathcal{R}} \subseteq Q_{\geq 2}$.

Sea $\varphi_0 : kQ \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ el único morfismo k-lineal tal que

$$\varphi_0(c) = \sum_{i=1}^{n} \pi(c_n \cdots c_{i+1}) \otimes c_i \otimes \pi(c_{i-1} \cdots c_1)$$

para $c \in Q_{\geq 0}$, $c = c_n \cdots c_1$ con $c_i \in Q_1$ para todo $i \in \{1, \dots, n\}$.

Dada una reducción básica $r = r_{a,s,c}$, sea $\varphi_1(r, -) : kQ \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ el único morfismo k-lineal tal que, dado $p \in Q_{\geq 0}$

$$\varphi_1(r, p) = \begin{cases} \pi(a) \otimes s \otimes \pi(c), & \text{si } p = asc, \\ 0 & \text{si } p \neq asc. \end{cases}$$

Si $r = (r_n, \dots, r_1)$ es una reducción, con r_i reducción básica para todo i, $1 \leq i \leq n$, denotamos $r' = (r_n, \dots, r_2)$ y definimos de manera recursiva el morfismo $\varphi_1(r, -)$ como el único morfismo k-lineal de kQ en $A \otimes_E k\mathcal{A}_1 \otimes_E A$ tal que

$$\varphi_1(r, p) = \varphi_1(r_1, p) + \varphi_1(r', r_1(p)).$$

Definimos $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ como

$$d_1(1 \otimes s \otimes 1) = \varphi_0(s) - \varphi_0(\beta(s)), \text{ para todo } s \in \mathcal{A}_1.$$

Sea $p \in \mathcal{A}_2$. Sabemos que p se escribe como $p = u_0 u_1 u_2$ y como $p = v_2 v_1 v_0$, igualdades que corresponden a las escrituras de p como 2-ambigüedad a izquierda y a derecha, respectivamente. Los elementos $u_0 u_1$ y $v_1 v_0$ pertenecen a $\mathcal{A}_1 = S$. Si $r = r_{a,\rho,c}$ es una reducción básica con $\rho = (s, f)$, decimos que r es una *reducción a izquierda* de p si $u_0 u_1 = s$ y decimos que r es una *reducción a derecha* de p si $s = v_1 v_0$. Toda reducción básica r tal que $r(p) \neq p$ es una reducción a izquierda o a derecha de p. Más generalmente, si $r = (r_n, \dots, r_1)$ es una reducción, decimos que r es una reducción a izquierda de p si r_1 es una reducción a izquierda de p. Análogamente definimos *reducción a derecha*.

Proposición (3.22). *Sean* $\{r^p\}_{p \in \mathcal{A}_2}$ *y* $\{t^p\}_{p \in \mathcal{A}_2}$ *conjuntos de reducciones tales que* $r^p(p)$ *y* $t^p(p)$ *pertenecen a* $k\mathcal{B}$, r^p *es una reducción a izquierda de* p *y* t^p *es una reducción a derecha de* p, *para todo* $p \in \mathcal{A}_2$. *Sea* $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ *el morfismo de* A-*bimódulos definido por*

$$d_1(1 \otimes s \otimes 1) = \varphi_0(s) - \varphi_0(\beta(s)), \textit{ para todo } s \in \mathcal{A}_1,$$

y $d_2 : A \otimes_E k\mathcal{A}_2 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ *el morfismo de* A-*bimódulos dado por*

$$d_2(1 \otimes p \otimes 1) = \varphi_1(t^p, p) - \varphi_1(r^p, p).$$

La sucesión

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

es exacta.

Capítulo 4: ejemplos

En este capítulo utilizamos los resultados obtenidos en nuestros teoremas para calcular resoluciones proyectivas explícitas de algunas familias de algebras.

Capítulo 5: álgebras *down-up*

En este capítulo estudiamos varios problemas relacionados con la familia de álgebras down-up, definida en [12]. Dicha familia se define de la siguiente manera. Sean k un cuerpo y $\alpha, \beta, \gamma \in k$, el álgebra *down-up de parámetros* α, β, γ se denota $A(\alpha, \beta, \gamma)$ y es el cociente de $k\langle d, u\rangle$ por el ideal bilátero generado por las relaciones

$$d^2u - \alpha dud - \beta ud^2 - \gamma d = 0,$$
$$du^2 - \alpha udu - \beta u^2d - \gamma u = 0.$$

Estas álgebras son 3-Koszul si y sólo si $\gamma = 0$ [9].

En [12] los autores plantean el problema de decidir qué álgebras pertenecen a la misma clase de isomorfismo, llamado *problema de isomorfismo*, y definen cuatro subfamilias de manera tal que álgebras en distintas familias no son isomorfas. Dichas familias están caracterizadas por las siguientes condiciones:

(a) $\gamma = 0$, $\alpha + \beta = 1$,

(b) $\gamma = 0$, $\alpha + \beta \neq 1$,

(c) $\gamma \neq 0$, $\alpha + \beta = 1$,

(d) $\gamma \neq 0$, $\alpha + \beta \neq 1$.

Como consecuencia de esta clasificación, el problema de isomorfismo se divide en cuatro problemas de isomorfismo, uno para cada subfamilia.

En [24] los autores prueban que el álgebra $A(\alpha, \beta, \gamma)$ es noetheriana si y sólo si $\beta \neq 0$, lo que implica que las álgebras $A(\alpha, \beta, \gamma)$ con $\beta \neq 0$ no son isomorfas a ninguna de las álgebras $A(\alpha', 0, \gamma')$. Por otro lado, en [16] los autores resuelven el problema de isomorfismo para las álgebras down-up noetherianas de tipo (a), (b) y (c) para todo cuerpo k y también para las álgebras down-up noetherianas de tipo (d) para cuerpos de característica cero.

Los resultados de nuestra investigación sobre las álgebras down-up son los siguientes.

1. Utilizando los métodos desarrollados en los capítulos anteriores, encontramos una resolución explícita de largo 3 para toda álgebra down-up $A(\alpha, \beta, \gamma)$. Dicha resolución coincide con la resolución ya conocida en los casos en que $A(\alpha, \beta, \gamma)$ es 3-Koszul.

2. Probamos que el álgebra $A(\alpha, \beta, \gamma)$ es monomial si y sólo si $(\alpha, \beta, \gamma) = (0, 0, 0)$. Recordamos que un álgebra es monomial si es isomorfa a un cociente de álgebras de caminos kQ/I con I un ideal bilátero generado por caminos. Como consecuencia, la resolución de Bardzell no se puede aplicar a las álgebras $A(\alpha, \beta, \gamma)$ con $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. La prueba de este resultado utiliza cálculos de invariantes homológicos que son posibles gracias a la descripción de la resolución proyectiva obtenida anteriormente.

3. Resolvimos el problema de isomorfismo para álgebras down-up no noetherianas de todos los tipos, para cuerpos de cualquier característica: el álgebra $A(\alpha, 0, \gamma)$ es isomorfa al álgebra $A(\alpha', 0, \gamma')$ si y sólo si $\alpha = \alpha'$ y existe $\lambda \in k$ tal que $\gamma = \lambda\gamma'$.

Résumé de la thèse

Dans le texte qui suit, nous résumons les principaux résultats contenus dans cette thèse. L'article [18] contient une partie de ce travail.

Chapitre 1 : Systèmes de réduction, ordres partiels et ambiguïtés

Dans ce chapitre nous définissons et étudions les outils que nous utilisons dans les chapitres suivants, afin de construire des résolutions projectives d'algèbres associatives. Il s'agit de *systèmes de réduction* et d'*ambiguïtés*.

Systèmes de réduction

Nous commençons par introduire quelques notations. Soient k un corps et Q un carquois avec un nombre fini de sommets. Notons $k^\times = k \setminus \{0\}$, $\mathbb{N}$ l'ensemble de nombres entiers positifs et $\mathbb{N}_0$ l'ensemble de nombres entiers non negatifs. Étant donné $n \in \mathbb{N}$, Q_n est l'ensemble des chemins de longueur n et $Q_{\geq n}$ l'ensemble des chemins de longueur au moins n. Si $c \in Q_n$, nous écrivons $|c| = n$. Si $a, b, p, q \in Q_{\geq n}$ sont tels que $q = apb$, on dit que p est un *diviseur* de q. Si $a \in Q_0$ on dit que p est un *diviseur à gauche* de q, et si $b \in Q_0$ on dit que p est un *diviseur à droite* de q. Soient $t, s : Q_1 \longrightarrow Q_0$ les applications qui associent à chaque flèche son *but* et sa *source*. Étant donnés $s \in Q_{\geq 0}$ et un élément $f = \sum_i \lambda_i c_i \in kQ$ tel que $c_i \in Q_{\geq 0}$ et $t(s) = t(c_i)$, $s(s) = s(c_i)$ pour chaque i, nous disons que f est *parallèle* à s. Soit $E := kQ_0$ la sous-algèbre de kQ engendrée par les sommets de Q.

Étant donné un anneau R, un R-module à gauche M est un ensemble $X \subseteq M$, siot $\langle X \rangle_R$ le R-sous-module à gauche de M engendré par X.

Nous utilisons la même terminologie que dans [3]. Soit $\mathcal{R}$ un sous-ensemble de $Q_{\geq 0} \times kQ$. On dit que $\mathcal{R}$ est *un système de réduction* si pour tout $(s, f) \in \mathcal{R}$, l'élément f est parallèle à s et $s \neq f$. Un triplet (a, ρ, c) où $\rho = (s, f) \in \mathcal{R}$ et $a, c \in Q_{\geq 0}$ sont tels que $asc \neq 0$ dans kQ s'apelle *réduction basique* et on la dénote $r_{a,\rho,c}$. Nous observons qu'une réduction basique $r_{a,\rho,c}$ définit un endomorphisme $r_{a,\rho,c} : kQ \longrightarrow kQ$ par

$r_{a,\rho,c}(asc) = afc$ et $r_{a,\rho,c}(q) = q$ pour tout chemin q différent de asc. Lorsqu'il n'y a pas de confusion possible, nous écrivons simplement $r_{a,s,c}$ au lieu de $r_{a,\rho,c}$. Une *réduction* est un n-uplet $r = (r_n, \dots, r_1)$ où $n \in \mathbb{N}$ et r_i est une réduction basique pour toute i. Toute réduction r définit un endomorphisme $r : kQ \longrightarrow kQ$ donné par la composition $r_n \circ \cdots \circ r_1$ des endomorphismes correspondants aux réductions basiques $r_n, \dots, r_1$.

Nous disons qu'un élément $x \in kQ$ est *irréductible* pour $\mathcal{R}$ si $r(x) = x$ pour toute réduction basique r. Un chemin $p \in Q_{\geq 0}$ est dit de *réduction finie* si pour toute suite infinie de réductions basiques $(r_i)_{i\in\mathbb{N}}$ il existe $n_0 \in \mathbb{N}$ tel que $r_n \circ \cdots \circ r_1(p) = r_{n_0} \circ \cdots \circ r_1(p)$ pour tout $n \geq n_0$. On dit qu'un chemin p est de *réduction unique* s'il est de réduction finie et en plus pour chaque couple de réductions r et r' tels que $r(p)$ et $r'(p)$ sont des éléments irréductibles, on a $r(p) = r'(p)$.

Un ordre partiel

Soient k un corps et Q un carquois comme plus haut. Soit $\mathcal{R}$ un système de réduction tel que tout chemin est de réduction finie.

Soit $x = \sum_{i=1}^{n} \lambda_i c_i \in kQ$ où $\lambda_1, \dots, \lambda_n \in k^\times$ et $c_1, \dots, c_n$ sont des chemins de longueur positive. Nous appelons $\{c_1, \dots, c_n\}$ le *support* de x et nous le dénotons $\mathsf{Su}(x)$.

Nous définissons une rélation binaire $\preceq$ sur l'ensemble $k^\times Q_{\geq 0} := \{\lambda p \in kQ : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$ comme la plus petite rélation réflexive et transitive qui vérifie $\lambda p \preceq \mu q$ s'il existe une réduction r tel que $r(\mu q) = \lambda p + x$ avec $p \notin \mathsf{Su}(x)$. Nous définissons $0 \preceq \lambda p$ pour tout $\lambda p \in k^\times Q_{\geq 0}$. Soient $x \in kQ$ et $\lambda p \in k^\times Q_{\geq 0}$. Si $x = \sum_{i=1}^{n} \lambda_i c_i$ où $\lambda_i \in k^\times$ et $\lambda_i p_i \preceq \lambda p$ pour tout i, nous écrivons $x \preceq \lambda p$. Par ailleurs, si $x \neq \lambda_i p_i$ pour tout i, nous écrivons $x \prec p$.

Lemme. *La rélation binaire $\preceq$ est une relation d'ordre partiel vérifiant la condition de chaîne descendante.*

La condition diamant

Soit I un idéal bilatère de kQ. Soit π la projection canonique $\pi : kQ \longrightarrow kQ/I$. Soit $\mathcal{R}$ un système de réduction. Nous disons que $\mathcal{R}$ vérifie la condition *diamant pour* I si

- l'idéal I est l'idéal bilatère engendré par l'ensemble $\{s - f\}_{(s,f)\in\mathcal{R}}$,
- tout chemin est de réduction unique et
- pour chaque $(s, f) \in \mathcal{R}$, l'élément f est irréductible.

Une conséquence du Lemme du diamant de Bergman [3] est la suivante.

Lemme. *Soit $\mathcal{R}$ un système de réduction vérifiant la condition diamant pour* I. *L'ensemble $\mathcal{B}$ de chemins irréductibles satisfait les propriétés suivantes.*

- *Si* $b \in \mathcal{B}$ *et* $q \in Q_{\geq 0}$ *est un diviseur de* b, *alors* $q \in \mathcal{B}$.
- $\pi(b) \neq \pi(b')$ *pour tout* $b, b' \in \mathcal{B}$ *tels que* $b \neq b'$.
- $\{\pi(b) : b \in \mathcal{B}\}$ *est une base de* kQ/I *comme* k*-espace vectoriel.*

Ce lemme est une des principales raisons pour lesquelles les systèmes de réduction qui vérifient la condition diamant sont utiles pour nos objectifs. La proposition suivante dit que tout idéal bilatère de kQ possède des systèmes de réduction de ce type.

Proposition. *Soit* I *un idéal bilatère de* kQ. *Il existe un système de réduction $\mathcal{R}$ vérifiant la condition diamant pour* I.

Soit $\mathcal{R}$ un système de réduction. Nous donnons par la suite quelques définitions supplémentaires.

- On définit une *ambiguïté d'inclusion* comme un 5-uplet $(\rho_1, \rho_2, a, b, c)$ où $\rho_1, \rho_2 \in \mathcal{R}$, $a, b, c \in Q_{\geq 0}$ sont tels que $\rho_1 = (abc, f_1)$ et $\rho_2 = (b, f_2)$ pour certains éléments $f_1, f_2 \in kQ$.
- On dénote par $S_{\mathcal{R}}$ l'ensemble $\{s \in Q_{\geq 0} :$ il existe $f \in kQ$ tel que $(s, f) \in \mathcal{R}\}$.

Le résultat suivant dit que tout système de réduction qui vérifie la condition diamant pour un certain idéal I peut se modifier pour qu'il n'ait pas des ambiguïtés d'inclusion.

Proposition. *Soient* I *un idéal bilatère de* kQ *et $\mathcal{R}$ un système de réduction vérifiant la condition diamant pour* I. *L'ensemble* $\mathcal{R} = \{(s, f) \in \mathcal{R} :$ *il n'existe pas une ambiguïté d'inclusion* $(\rho_1, \rho_2, a, b, c)$ *tel que* $abc = s\}$ *est un système de réduction qui satisfait la condition diamant pour* I *et il n'a pas des ambiguïtés d'inclusion.*

De plus, si $\mathcal{R}$ est un système de réduction, sans ambiguïtés d'inclusion et qui vérifie la condition diamant pour un idéal I, alors il peut se modifier pour qu'il vérifie $S_{\mathcal{R}} \subseteq Q_2$.

Ambiguïtés

Soient I un idéal bilatère de kQ et $\mathcal{R}$ un système de réduction vérifiant la condition diamant, sans ambiguïtés d'inclusion, et tel que $S_{\mathcal{R}} \subseteq Q_2$. Nous rapellons la défintion d'une n-ambiguïté. Cette définition peut se trouver dans les articles [1], [2], [4] et [32].

Définition. Étant donnés $n \geq 2$ et $p \in Q_{\geq 0}$,

1. le chemin p s'apelle une n-*ambiguïté à gauche* s'il existe $u_0 \in Q_1$ et des chemins irréductibles $u_1, \ldots, u_n$ tels que

   (a) $p = u_0 u_1 \cdots u_n$,

   (b) pour tout i, le chemin $u_i u_{i+1}$ n'est pas irréductible mais $u_i d$ est irréductible pour tout diviseur à gauche d de u_{i+1}, différent de u_{i+1}.

2. le chemin p s'apelle une n-*ambiguïté à droite* s'il existe $v_0 \in Q_1$ et des chemins irréductibles $v_1, \ldots, v_n$ tels que

   (a) $p = v_n \cdots v_0$,

   (b) pour tout i, le chemin $v_{i+1} v_i$ n'est pas irréductible mais $d v_i$ est irréductible pour tout diviseur à droite d de v_{i+1}, différent de v_{i+1}.

Soient $\mathcal{A}_{-1} := Q_0$, $\mathcal{A}_0 := Q_1, \mathcal{A}_1 := S_{\mathcal{R}}$ et pour tout $n \geq 2$, nous définissons $\mathcal{A}_n$ et $\mathcal{A}'_n$ les ensembles de n-ambiguïtés à gauche et à droite, respectivement.

Proposición. *Pour tout $n \geq 2$ on a $\mathcal{A}_n = \mathcal{A}'_n$. De plus, $\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ si n et m sont différents.*

Chapitre 2 : Des travaux précédents d'Anick, Green et Bardzell

Ce chapitre contient un résumé des travaux précédents d'Anick, Green et Bardzell sur la construction de résolutions projectives. Voir [1], [2], [4] y [32].

La résolution d'Anick

Soit A une k-algèbre presentée comme $A = k\langle X \rangle / I$, où X est un ensemble de générateurs de A et I est un idéal de $k\langle X \rangle$. Soient $\leq$ un ordre partiel sur X et $\omega : X \longrightarrow \mathbb{N}$ une fonction. Ceci définit un ordre total sur l'ensemble des monômes sur X, apellé l'ordre *deglex*, et on le dénote par $\leq_\omega$. Il s'agit d'un ordre lexicographique avec poids. Anick a prouvé qu'il existe des sous-ensembles $\mathcal{A}_n$ de $\langle X \rangle$, où $n \geq 2$, tels que pour tout A-module simple T, il existe une résolution libre de T de la forme

$$0 \longleftarrow T \xleftarrow{\epsilon} A \xleftarrow{d_0} kX \otimes_k A \xleftarrow{d_1} kS \otimes_k A \xleftarrow{d_2} k\mathcal{A}_2 \otimes_k A \xleftarrow{d_3} \cdots,$$

où S est un ensemble minimal de générateurs de I et kY dénote le k-espace vectoriel engendré par un ensemble de chemins Y. Avec notre notation, les ensembles $\mathcal{A}_n$ correspondent aux n-ambiguïtés d'un système de réduction qu'on construit à partir de $\leq_\omega$. La seule information qu'on a sur les différentielles d_i est la suivante. L'ordre sur les monômes $\leq_\omega$ peut se prolonger à un ordre total sur l'ensemble de tenseurs élémentaires de manière que pour tout n, les terms de $d_n(u_0 \cdots u_n \otimes 1) - u_0 \cdots u_{n-1} \otimes u_n$ sont plus petits que $u_0 \cdots u_{n-1} \otimes u_n$, pour tout $p = u_0 \cdots u_n \in \mathcal{A}_n$.

La résolution d'Anick-Green est une généralisation de ce résultat au cadre des algebres de chemins.

La résolution de Bardzell

Soit Q un carquois avec un nombre fini de sommets. Lorsqu'un idéal bilatère de kQ est engendré par des chemins de longueur au moins 2, l'idéal est dit *monomial*. On dit qu'une k-algèbre A est *monomiale* s'il existe un carquois Q et un idéal monomial I de kQ tels que $A \cong kQ/I$.

Soit $A = kQ/I$ où I est un idéal monomial. Soit $\pi : kQ \longrightarrow A$ la projection canonique. Soit S l'ensemble de chemins $s \in I$ tels que $s' \notin I$ pour tout diviseur s' de s, différent de s. Nous observons que l'idéal I est engendré par l'ensemble S. Bardzell a prouvé qu'il existe des sous-ensembles $\mathcal{A}_n$ de $Q_{\geq 0}$, où $n \geq 2$, donnant lieu à une résolution projective de A en tant que bimodule. Avec notre notation, les ensembles $\mathcal{A}_n$ correspondant aux ensembles de n-ambiguïtés du système de réduction $\mathcal{R} = \{(s, 0) : s \in S\}$. La résolution de Bardzell est la suivante.

$$\cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

$$A \otimes_E A \overset{\cong}{\longrightarrow} A \otimes_E k\mathcal{A}_{-1} \otimes_E A$$

où $\mathcal{A}_{-1} = Q_0$, $\mathcal{A}_0 = Q_1$, $\mathcal{A}_1 = S$, et

1. $d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ pour $n \geq 0$,

2. $d_{-1}(a \otimes b) = ab$ est la multiplication de A,

3. si n est pair, $q \in \mathcal{A}_n$ et $q = u_0 \cdots u_n = v_n \cdots v_0$ sont, respectivement, les factorisations de q en tant que n-ambiguïté à gauche et à droite,

$$d_n(1 \otimes q \otimes 1) = \pi(v_n) \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes \pi(u_n),$$

4. si n es impair et $q \in \mathcal{A}_n$,

$$d_n(1 \otimes q \otimes 1) = \sum_{\substack{apc=q \\ p \in \mathcal{A}_{n-1}, a,c \in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c).$$

Cette résolution a une contraction d'homotopie $s_\bullet$ donnée par les formules suivantes.

Pour $n = -1$, $s_{-1} : A \longrightarrow kQ \otimes_E k\mathcal{A}_{-1} \otimes_E A$ est le morphisme de $kQ - E$-bimodules défini par $s_{-1}(a) = a \otimes 1$, où $a \in kQ$. Pour $n \in \mathbb{N}_0$, $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ est

$$s_n(1 \otimes q \otimes \pi(b)) = (-1)^{n+1} \sum_{\substack{apc=qb \\ p\in\mathcal{A}_n, a,c\in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c),$$

où $b \in \mathcal{B}$ et $q \in \mathcal{A}_{n-1}$. Cette contraction d'homotopie a été obtenue par Sköldberg dans l'article [32].

Chapitre 3 : Construction de résolutions projectives utilisant des ambiguïtés

Il s'agit du chapitre central de la thèse, où nous énonçons et prouvons nos principaux résultats.

Soient Q un carquois e I un idéal bilatère de kQ. Nous apellons $A = kQ/I$ et $\pi : kQ \longrightarrow A$ la projection canonique. Soit $\mathcal{R}$ un système de réduction vérifiant la condition diamant, sans ambiguïtés d'inclusion, tel que $S_{\mathcal{R}} \subseteq Q_{\geq 2}$. Nous dénotons $S = S_{\mathcal{R}}$ et $\mathcal{B} = \{p \in Q_{\geq 0} : p \text{ est irréductible}\}$.

Nous associons à A la k-algèbre monomiale $A_S := kQ/\langle S\rangle$. Soit $\pi' : kQ \longrightarrow A_S$ la projection canonique. Nous observons que l'ensemble de n-ambiguïtés du système de réduction $\mathcal{R}' = \{(s, 0) : s \in S\}$ est égale à l'ensemble de n-ambiguïtés de $\mathcal{R}$. Donc, l'ensemble des chemins irréductibles pour $\mathcal{R}$ est égale à l'ensemble des chemins irréductibles de $\mathcal{R}'$. Il y a des morphismses de k-espaces vectoriels $i : A \longrightarrow kQ$ et $i' : A_S \longrightarrow kQ$ tels que $i(\pi(b)) = b$ et $i'(\pi'(b)) = b$ pour tout $b \in \mathcal{B}$. Nous définissons $\beta = i \circ \pi$ et $\beta' = i' \circ \pi'$. Pour chaque $n \geq -1$, nous considérons les applications k-linéaires suivantes.

$$\begin{aligned} \pi_n &:= \pi \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi, & \pi'_n &:= \pi' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi', \\ i_n &:= i \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i, & i'_n &:= i' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i', \\ \beta_n &:= i_n \circ \pi_n, & \beta'_n &:= i'_n \circ \pi'_n. \end{aligned}$$

Nous nous basons ensuite sur les formules des différentielles de la résolution de Bardzell de A_S pour construir des morphismes de A-bimodules $\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$. Pour faire cela, nous commençons par construir les morphismes f_n et S_n suivantes. Nous observons que l'ensemble $\{a \otimes q \otimes c : a, c \in Q_{\geq 0}, q \in \mathcal{A}_n, aqc \neq 0 \in kQ\}$ est une base de $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ en tant que k-espace vectoriel. Nous considérons le diagramme suivant.

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{f_2} & kQ \otimes_E k\mathcal{A}_1 \otimes_E kQ & \xrightarrow{f_1} & kQ \otimes_E k\mathcal{A}_0 \otimes_E A & \xrightarrow{f_0} & kQ \otimes_E kQ & \xrightarrow{f_{-1}} & kQ & \longrightarrow 0 \\
\xleftarrow{S_2} & & \xleftarrow{S_1} & & \xleftarrow{S_0} & \downarrow \cong & \xleftarrow{S_{-1}} & & \\
& & & & & kQ \otimes_E k\mathcal{A}_{-1} \otimes_E kQ & & &
\end{array}
$$

où

1. $f_{-1}(a \otimes b) = ab$,
2. si n est pair, alors nous définissons f_n comme la seule application k-linéaire tel que pour tous $a, c \in Q_{\geq 0}$, $q \in \mathcal{A}_n$ tels que $aqc \neq 0$ dans kQ et $q = u_0 \cdots u_n = v_n \cdots v_0$ sont respectivement les factorisations de q en tant que n-ambiguïté à gauche et à droite,

   $$f_n(a \otimes q \otimes c) = av_n \otimes v_{n-1} \cdots v_0 \otimes c - a \otimes u_0 \cdots u_{n-1} \otimes u_n c,$$

   et nous observons que cette application est en fait un morphisme de kQ-bimodules.
3. Si n es impair, f_n es la seule application k-linéaire tel que pour tous $a, c \in Q_{\geq 0}, q \in \mathcal{A}_n$ comme plus haut,

   $$f_n(a \otimes q \otimes c) = \sum_{\substack{a'pc'=q \\ p \in \mathcal{A}_{n-1}, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'c,$$

   et il résulte un morphisme de kQ-bimodules.
4. Nous définissons $S_{-1}(x) = x \otimes 1$ et si $n \geq 0$, S_n est la seule application k-linéaire tel que pour tous $a, c \in Q_{\geq 0}, q \in \mathcal{A}_{n-1}$,

   $$S_n(a \otimes q \otimes c) = (-1)^{n+1} \sum_{\substack{a'pc'=qc \\ p \in \mathcal{A}_n, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'.$$

   Nous observons que S_n est un morphisme de $kQ - E$-bimodules pour tout $n \geq -1$.

Les morphismes f_n définissent des morphismes de A-bimodules $\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ et des morphismes de A_S-E-bimodules $\delta'_n : A_S \otimes_E k\mathcal{A}_n \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S$ comme suit

$$\delta_n := \pi_{n-1} \circ f_n \circ i_n,$$
$$\delta'_n := \pi'_{n-1} \circ f_n \circ i'_n.$$

Pour $n = -1$ nous interprétons ces formules comme $\delta_{-1} := \pi \circ f_{-1} \circ i_{-1}$ et $\delta'_{-1} := \pi' \circ f_{-1} \circ i'_{-1}$. Nous observons que ce sont les multiplications de A et A_S, respectivement.

La résolution de Bardzell de A_S est la suivante

$$\cdots \xrightarrow{\delta'_2} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E A_S \xrightarrow{\delta'_{-1}} A_S \longrightarrow 0\,,$$

Les morphismes S_n définissent des morphismes de $A - E$ bimodules $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$, et des morphismes de $A_S - E$-bimodules $s'_n : A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_n \otimes_E A_S$ de la manière suivante.

$$s_n := \pi_n \circ S_n \circ i_{n-1},$$
$$s'_n := \pi'_n \circ S_n \circ i'_{n-1}.$$

Pour $n = -1$ nous interprétons ces définitions comme $s_{-1} := \pi_{-1} \circ S_{-1} \circ i$ et $s'_{-1} := \pi'_{-1} \circ S_{-1} \circ i'$. Les morphismes s'_n sont la contraction d'homotopie de Sköldberg pour la résolution de Bardzell.

Nous rappelons que $k^\times Q_{\geq 0} := \{\lambda p \in kQ : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$. Pour chaque $n \geq -1$ et $\mu q \in k^\times Q_{\geq 0}$, nous définissons les sous-ensembles de $A \otimes_E k\mathcal{A}_n \otimes_E A$ suivants.

- $\overline{\mathcal{L}}_n^{\preceq}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda b p b' \preceq \mu q\}$,
- $\overline{\mathcal{L}}_n^{\prec}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda b p b' \prec \mu q\}$.

Nous sommes prêts pour présenter nos théorèmes. Le premièr théorème dit que si $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ sont des morphismes de A-bimodules formant un complexe tel que ses formules sont données par certaines modifications des formules des différentielles de la résolution de Bardzell de A_S, alors ce complexe est nécessairement exacte. Le deuxième théorème dit que ce type de résolutions existent dans tous les cas.

Théorème (3.5). *Posons* $d_{-1} := \delta_{-1}$ *et* $d_0 := \delta_0$. *Étant donné* $N \in \mathbb{N}_0$ *et des morphismes de* A*-bimodules* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$, *où* $1 \leq i \leq N$, *vérifiant les conditions suivantes*

1. $d_{i-1} \circ d_i = 0$ *pour tout* i, $1 \leq i \leq N$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}_{i-1}^{\prec}(q) \rangle_k$ *pour tout* $i \in \{1, \ldots, N\}$ *et pour tout* $q \in \mathcal{A}_i$,

alors le complexe

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

est exacte.

Théorème (3.6). *Il existent des morphismes de* A*-bimodules* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$*, où* $i \in \mathbb{N}_0$ *et* $d_{-1} : A \otimes_E A \longrightarrow A$ *tels que*

1. $d_{i-1} \circ d_i = 0$*, pour tout* $i \in \mathbb{N}_0$*,*
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q) \rangle_{\mathbb{Z}}$ *pour tout* $i \geq -1$ *et* $q \in \mathcal{A}_i$*.*

Morphismes en degrés 0, 1 et 2

Nous pouvons donner une description des différentielles pour ces résolutions en degrés $0, 1$ et 2. Comme auparavant, soient $A = kQ/I$ une algèbre et $\mathcal{R}$ un système de réduction vérifiant la condition diamant, sans ambiguïtés d'inclusion, tel que $S_{\mathcal{R}} \subseteq Q_{\geq 2}$.

Soit $\varphi_0 : kQ \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ la seule application k-linéaire tel que

$$\varphi_0(c) = \sum_{i=1}^{n} \pi(c_n \cdots c_{i+1}) \otimes c_i \otimes \pi(c_{i-1} \cdots c_1)$$

où $c \in Q_{\geq 0}$, $c = c_n \cdots c_1$ avec $c_i \in Q_1$ pour tout $i \in \{1, \dots, n\}$.

Étant donnée une réduction basique $r = r_{a,s,c}$, soit $\varphi_1(r, -) : kQ \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ la seule application k-linéaire tel que pour $p \in Q_{\geq 0}$

$$\varphi_1(r, p) = \begin{cases} \pi(a) \otimes s \otimes \pi(c), & \text{si } p = asc, \\ 0 & \text{si } p \neq asc. \end{cases}$$

Si $r = (r_n, \dots, r_1)$ est une réduction, où r_i est une réduction basique pour tout i, $1 \leq i \leq n$, nous posons $r' = (r_n, \dots, r_2)$ et nous définissons le morphisme $\varphi_1(r, -)$, de manière récursive, comme la seule application k-linéaire $\varphi_1(r, -) : kQ \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ telle que

$$\varphi_1(r, p) = \varphi_1(r_1, p) + \varphi_1(r', r_1(p)).$$

Nous définissons $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ comme suit

$$d_1(1 \otimes s \otimes 1) = \varphi_0(s) - \varphi_0(\beta(s)), \text{ pour tout } s \in \mathcal{A}_1.$$

Soit $p \in \mathcal{A}_2$. Nous savons que p s'écrit comme $p = u_0u_1u_2$ et comme $p = v_2v_1v_0$. Ces égalités correspondent respectivement aux écritures de p en tant que 2-ambiguïté à gauche et à droite. Les éléments u_0u_1 et v_1v_0 appartiennent à $\mathcal{A}_1 = S$. Si $r = r_{a,\rho,c}$ est une réduction basique avec $\rho = (s, f)$, nous disons que r est une *réduction à gauche* de p si $u_0u_1 = s$ et nous disons que r est une *réduction à droite* de p si $s = v_1v_0$. Toute réduction basique r tel que $r(p) \neq p$ est une réduction à gauche ou à droite de p. Plus généralement, si $r = (r_n, \dots, r_1)$ est une réduction, nous disons que r est une réduction à gauche de p si r_1 est une réduction à gauche de p. Similairement, nous définissons une *réduction à droite* de p.

Proposition (3.22). *Soient* $\{r^p\}_{p\in\mathcal{A}_2}$ *et* $\{t^p\}_{p\in\mathcal{A}_2}$ *des ensembles de réductions tels que* $r^p(p)$ *et* $t^p(p)$ *appartiennent à* $k\mathcal{B}$ *et pour tout*$p \in \mathcal{A}_2$*, la réduction* r^p *est une réduction à gauche de* p *et* t^p *est une réduction à droite de* p. *Soit* $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ *le morphisme de* A*-bimodules défini par*

$$d_1(1 \otimes s \otimes 1) = \varphi_0(s) - \varphi_0(\beta(s)), \text{ pour tout } s \in \mathcal{A}_1,$$

et $d_2 : A \otimes_E k\mathcal{A}_2 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ *le morphisme de* A*-bimodules donné par*

$$d_2(1 \otimes p \otimes 1) = \varphi_1(t^p, p) - \varphi_1(r^p, p).$$

La suite

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

est exacte.

Chapitre 4 : Examples

Dans ce chapitre nous utilisons les résultats obtenus dans nos théorèmes pour calculer des résolutions projectives de quelques familles d'algèbres.

Chapitre 5 : Algèbres *down-up*

Dans ce chapitre nous étudions des différents problèmes lies à la famille d'algèbres down-up, définie en [12]. On définit cette famille comme suit. Soit k un corps et $\alpha, \beta, \gamma \in k$, l'algèbre *down-up à paramètres* α, β, γ se dénote $A(\alpha, \beta, \gamma)$ et se définit comme le quotient de $k\langle d, u\rangle$ par l'idéal bilatère engendré par les rélations suivantes

$$d^2u - \alpha dud - \beta ud^2 - \gamma d = 0,$$
$$du^2 - \alpha udu - \beta u^2d - \gamma u = 0.$$

Ces algèbres sont 3-Koszul si et seulement si $\gamma = 0$ [9].

Dans l'article [12], les auteurs posent le problème de classer, à isomorphisme près, les algèbres *down-up*. On parle du *problème de l'isomorphisme*. Ils définissent quatre sous-familles d'algèbres de manière que des algèbres dans des différentes familles ne sont pas isomorphes. Ces familles sont définis par les conditions suivantes:

(a) $\gamma = 0$, $\alpha + \beta = 1$,

(b) $\gamma = 0$, $\alpha + \beta \neq 1$,

(c) $\gamma \neq 0$, $\alpha + \beta = 1$,

(d) $\gamma \neq 0$, $\alpha + \beta \neq 1$.

Comme consequence de cette classification, le problème de l'isomorphisme se divise entre quatre sous-problèmes. Un problème pour chaque sous-famille.

Dans [24] les auteurs démontrent que l'algèbre $A(\alpha, \beta, \gamma)$ est noethérienne si et seulement si $\beta \neq 0$, ce que implique que les algèbres $A(\alpha, \beta, \gamma)$ avec $\beta \neq 0$ ne sont isomorphes à aucune des algèbres $A(\alpha', 0, \gamma')$. Par ailleurs, dans l'article [16] les auteurs résolvent le problème de l'isomorphisme pour les algèbres down-up noethériennes de type (a), (b) y (c) pour tout corps k, ainsi que pour les algèbres down-up noethériennes de type (d) sur des corps de caractéristique zero.

Nous obtenons les résultats suivants sur la famille d'algèbres down-up.

1. En utilisant les méthodes développées dans les chapitres antérieurs, nous trouvons une résolution de longueur 3 pour toute algèbre down-up $A(\alpha, \beta, \gamma)$. Cette résolution coïncide avec la résolution déjà connue lorsque $A(\alpha, \beta, \gamma)$ est 3-Koszul.

2. Nous montrons que l'algèbre $A(\alpha, \beta, \gamma)$ est monomiale si et seulement si $(\alpha, \beta, \gamma) = (0, 0, 0)$. Comme conséquence, la résolution de Bardzell ne peut pas s'apliquer aux algèbres $A(\alpha, \beta, \gamma)$ avec $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. La preuve de ce résultat s'appuie sur des calculs des invariantes homologiques qui sont rendus possibles grâce à la description de la résolution projective obtenue antérieurement.

3. Nous donnons une solution au problème de l'isomorphisme pour les algèbres down-up non-noethériennes de tous les types sur des corps de caractéristique quelconque: l'algèbre $A(\alpha, 0, \gamma)$ est isomorphe à l'algèbre $A(\alpha', 0, \gamma')$ si et seulement si $\alpha = \alpha'$ et il existe $\lambda \in k$ tel que $\gamma = \lambda\gamma'$.

Chapter 1

Reduction systems, partial orders and ambiguities

In this chapter we define and study the main tools that we will need in rest of the thesis. These are *reduction systems* and *ambiguities*. In Section 1.1 we set some basic notation and recall the terminology of *Reduction systems* from [3]. We prove that there are partial orders attached to reduction systems and these will be very important in the following chapters, since they permit to make inductive arguments. Also, the reduction systems that we will use to construct resolutions of algebras verify a special condition, namely *the Diamond condition*. Sections 1.2 and 1.3 are devoted to study these kind of reduction systems. In Section 1.4 we define and study the sets of *ambiguities*.

1.1 Reduction systems and partial orders

Let k be a field and Q a quiver with a finite set of vertices. Given $n \in \mathbb{N}$, Q_n denotes the set of paths of length n in Q and $Q_{\geq n}$ the set of paths of length at least n, that is, $Q_{\geq n} = \bigcup_{i \geq n} Q_i$. Whenever $c \in Q_n$, we will write $|c| = n$. If $a, b, p, q \in Q_{\geq 0}$ are such that $q = apb$, we say that p is a *divisor* of q; if, moreover, $a \in Q_0$, we say that p is a *left divisor* of q and analogously for $b \in Q_0$ and *right divisor*. We denote $t, s : Q_1 \longrightarrow Q_0$ the usual source and target functions. Given $s \in Q_{\geq 0}$ and a finite sum $f = \sum_i \lambda_i c_i \in kQ$ such that $c_i \in Q_{\geq 0}$ and $t(s) = t(c_i)$, $s(s) = s(c_i)$ for all i, we say that f is *parallel* to s. Let $E := kQ_0$ be the subalgebra of the path algebra generated by the vertices of Q.

Observe that for a set X, the free algebra $k\langle X\rangle$ is isomorphic to the path algebra kQ where Q is the quiver with one vertex and an arrow for each element of X. Under this identification, monomials in X correspond to paths in kQ.

Given a ring R, a left R-module M and a set $X \subseteq M$, we denote $\langle X\rangle_R$ the left

R-submodule of M spanned by X.

We recall some terminology from [3] that we will use. A set $\mathcal{R} \subseteq Q_{\geq 0} \times kQ$ is called a *reduction system* if for all $(s, f) \in \mathcal{R}$, the element f is parallel to s and $s \neq f$.

Given $\rho = (s, f) \in \mathcal{R}$ and $a, c \in Q_{\geq 0}$ such that $asc \neq 0$ in kQ, we will call the triple (a, ρ, c) a *basic reduction* and write it $r_{a,\rho,c}$. Note that $r_{a,\rho,c}$ determines an E-bimodule endomorphism $r_{a,\rho,c} : kQ \longrightarrow kQ$ such that $r_{a,\rho,c}(asc) = afc$ and $r_{a,\rho,c}(q) = q$ for all $q \neq asc$. If it is not ambiguous, we will write $r_{a,s,c}$ instead of $r_{a,\rho,c}$.

A *reduction* is an n-tuple $(r_n, \dots, r_1)$ where $n \in \mathbb{N}$ and r_i is a basic reduction for $1 \leq i \leq n$. As before, a reduction $r = (r_n, \dots, r_1)$ determines an E-bimodule endomorphism of kQ, the composition $r_n \circ \cdots \circ r_1$ of the endomorphisms corresponding to the basic reductions $r_n, \dots, r_1$.

An element $x \in kQ$ is said to be *irreducible* for $\mathcal{R}$ if $r(x) = x$ for all basic reductions r. We will omit mentioning the reduction system whenever it is clear from the context. A path $p \in Q_{\geq 0}$ will be called *reduction-finite* if for any infinite sequence of basic reductions $(r_i)_{i \in \mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $r_n \circ \cdots \circ r_1(p) = r_{n_0} \circ \cdots \circ r_1(p)$. Moreover, the path p will be called *reduction-unique* if it is reduction-finite and for any two reductions r and r' such that $r(p)$ and $r'(p)$ are both irreducible, the equality $r(p) = r'(p)$ holds.

Let $\mathcal{R}$ be a reduction system.

- An *inclusion ambiguity* is a 5-tuple $(\rho_1, \rho_2, a, b, c)$ with $\rho_1, \rho_2 \in \mathcal{R}$, $a, b, c \in Q_{\geq 0}$, such that $\rho_1 = (abc, f_1)$ and $\rho_2 = (b, f_2)$ for $f_1, f_2 \in kQ$. An inclusion ambiguity $(\rho_1, \rho_2, a, b, c)$ is said to be *resolvable* if there exist reductions $r = (r_n, \dots, r_1)$ and $t = (t_m, \dots, t_1)$ such that $r_1 = r_{1,\rho_1,1}$, $t_1 = r_{a,\rho_2,c}$ and $r(abc) = t(abc)$.

- An *overlap ambiguity* is a 5-tuple $(\rho_1, \rho_2, a, b, c)$ with $\rho_1, \rho_2 \in \mathcal{R}, a, c \in Q_{\geq 0}, b \in Q_{\geq 1}$ such that $\rho_1 = (ab, f_1)$ and $\rho_2 = (bc, f_2)$, for $f_1, f_2 \in kQ$. An overlap ambiguity $(\rho_1, \rho_2, a, b, c)$ is said to be resolvable if there exist reductions $r = (r_n, \dots, r_1)$ and $t = (t_m, \dots, t_1)$ such that $r_1 = r_{1,\rho_1,c}$, $t_1 = r_{a,\rho_2,1}$ and $r(abc) = t(abc)$.

- An overlap ambiguity $(\rho_1, \rho_2, a, b, c)$ is said to be a *minimal overlap ambiguity* if there is no other overlap ambiguity $(\rho_1', \rho_2', x, y, z)$ with xyz a proper divisor of abc.

- An *ambiguity* is a 5-tuple $(\rho_1, \rho_2, a, b, c)$ that is either an inclusion ambiguity or a minimal overlap ambiguity.

Definition 1.1. For any element $x = \sum_{i=1}^{n} \lambda_i c_i \in kQ$ with $\lambda_1, \dots, \lambda_n \in k^{\times}$ and $c_1, \dots, c_n \in Q_{\geq 0}$, the *support of* x is $\{c_1, \dots, c_n\}$ and we will denote it $\mathsf{Su}(x)$.

In the following chapters we will face the need to control how elements multiply in algebras of the form kQ/I, where I is a two-sided ideal of kQ. One way to do this is to find a k-basis $\mathcal{B}$ of kQ/I and write in terms of this basis the product of every pair of elements of $\mathcal{B}$. In [3], Bergman studies how to construct k-bases of algebras given by generators and relations using reduction systems, and proves the following theorem, which can also be applied to the setting of path algebras over a quiver Q.

Theorem 1.2 ([3]). *Let X be a set, $\mathcal{R}$ a reduction system for $k\langle X\rangle$ and $\leq$ a partial order on the set of paths $\langle X\rangle$ such that it satisfies the descending chain condition and for all $(s, f) \in \mathcal{R}$ and $c \in \mathsf{Su}(f)$, the inequality $s > c$ holds. Furthermore, suppose $b < b'$ implies $abc < ab'c$ for all $b, b', a, c \in \langle X\rangle$. The following conditions are equivalent.*

1. *All ambiguities of $\mathcal{R}$ are resolvable.*
2. *All elements of $k\langle X\rangle$ are reduction-unique.*
3. *The set of classes of the irreducible paths is a k-basis of kQ/I, where I is the two sided ideal generated by the set $\{s - f : (s, f) \in \mathcal{R}\}$.*

In [3], Section 5.4, Bergman states without a proof that if the reduction system $\mathcal{R}$ satisfies that every path is reduction-finite, then $\mathcal{R}$ induces a partial order $\rightsquigarrow$ on the set $Q_{\geq 0}$ that satisfies the hypothesis of Theorem 1.2. A slight modification of this partial order is key to our inductive process in Chapter 3. We now construct the binary relation $\rightsquigarrow$ and prove that it is a partial order.

Given $p, q \in Q_{\geq 0}$ we write $q \rightsquigarrow p$ if there exist $n \in \mathbb{N}$, basic reductions $r_1, \ldots, r_n$ and paths $p_1, \ldots p_n$, such that $p_1 = q$, $p_n = p$, and for all $i = 1, \ldots, n-1$, $p_{i+1} \in \mathsf{Su}(r_i(p_i))$.

Lemma 1.3. *Suppose that $\mathcal{R}$ is a reduction system for which every path is reduction-finite.*

1. *If p is a path and r a basic reduction such that $p \in \mathsf{Su}(r(p))$, then $r(p) = p$.*
2. *The binary relation $\rightsquigarrow$ is an order on the set $Q_{\geq 0}$ which is compatible with concatenation, that is, $\rightsquigarrow$ satisfies that $q \rightsquigarrow p$ implies $aqc \rightsquigarrow apc$ for all $a, c \in Q_{\geq 0}$ such that $apc \neq 0$ and $aqc \neq 0$ in kQ.*
3. *The binary relation $\rightsquigarrow$ satisfies the descending chain condition.*

Proof. (1) The hypothesis means that $r(p) = \lambda p + x$ with $\lambda \in k^\times$ and $p \notin \mathsf{Su}(x)$. If $x \neq 0$ or $\lambda \neq 1$, then r acts nontrivially on p and so it acts trivially on x. Since the sequence of reductions $(r, r, \cdots)$ stabilizes when acting on p, there exists $k \in \mathbb{N}$ such that $\lambda^k p + (\sum_{i=0}^{k-1} \lambda^i)x = r^k(p) = r^{k+1}(p) = \lambda^{k+1} p + (\sum_{i=0}^{k} \lambda^i)x$. As a consequence, $\lambda = 1$ and $x = 0$.

(2) By definition, the relation $\rightsquigarrow$ is transitive and reflexive. Let us suppose that it is not antisymmetric, so that there exist $n \in \mathbb{N}$, paths $p_1, \ldots, p_{n+1}$ and basic reductions $r_1, \ldots, r_n$ such that $p_{i+1} \in r_i(p_i)$ for $1 \leq i \leq n$ and $p_{n+1} = p_1$. Suppose that n is minimal. There exist $x_1, \ldots, x_n \in kQ$ and $\lambda_1, \ldots, \lambda_n \in k^\times$ such that $r_i(p_i) = \lambda_i p_{i+1} + x_i$

with $p_{i+1} \notin x_i$. Notice that since n is minimal, $r_i(p_i) \neq p_i$ and then r_i acts trivially on every path different from p_i, for all i.

Let us see that

$$p_i \notin \mathsf{Su}(x_j) \text{ for all } i \neq j.$$

Since the sequence $p_1, \dots, p_{n+1} = p_1$ is cyclic, it is enough to prove that $p_1 \notin \mathsf{Su}(x_j)$ for all j. Suppose that $p_1 \in \mathsf{Su}(x_j)$ for some $j \in \{1, \dots, n\}$. Since $p_{i+1} \notin \mathsf{Su}(x_i)$ for all i and $p_{n+1} = p_1$, it follows that $j \neq n$, and by part (i) $j \neq 1$. Let $u_k = p_k$ and $t_k = r_k$ for $1 \leq k \leq j$ and $u_{j+1} = p_1$. Notice that $u_{k+1} \in t_k(u_k)$ for $1 \leq k \leq j$ and $u_{j+1} = u_1$. Since $j < n$ this contradicts the choice of n. It follows that

$$p_i \notin \mathsf{Su}(x_j) \text{ for all } i, j.$$

This implies $r_n \circ \cdots \circ r_1(p_1) = \lambda p_1 + x$ for some $\lambda \in k^\times$ and $x \in kQ$ with $p_i \notin \mathsf{Su}(x)$ for all i. Now, define inductively for $i > n$, $r_i := r_{i-n}$. The sequence $(r_i)_{i \in \mathbb{N}}$ acting on p_1 never stabilizes, which contradicts the reduction-finiteness of the reduction system $\mathcal{R}$.

Let us see that $\rightsquigarrow$ is compatible with concatenation. By transitivity, it is enough to see that if p, q, a, c are paths and $r_{a',\rho,c'}$ is a basic reduction such that $apc \neq 0$ and $p \in \mathsf{Su}(r_{a',\rho,c'}(q))$, then $aqc \rightsquigarrow apc$. Write $\rho = (s, f)$. If $p = q$ there is nothing to prove. Suppose $p \neq q$. Since $p \in \mathsf{Su}(r_{a',\rho,c'}(q))$, we obtain that $r_{a',\rho,c'}(q) \neq q$. The only path on which the basic reduction $r_{a',\rho,c'}$ acts nontrivially is $a'sc'$, and so $q = a'sc'$. Recall that $r_{a',\rho,c'}(q) = a'fc'$. The fact that $p \in \mathsf{Su}(a'fc')$ implies that $apc \in \mathsf{Su}(aa'fc'c)$. Consider the basic reduction $\tilde{r} = r_{aa',\rho,c'c}$. Then $\tilde{r}(q) = aa'fc'c$ and we obtain that $apc \in \mathsf{Su}(\tilde{r}(aqc))$. Therefore $aqc \rightsquigarrow apc$.

(3) Suppose not, so that there is a sequence $(p_i)_{i \in \mathbb{N}}$ of paths and a sequence of basic reductions $(t_i)_{i \in \mathbb{N}}$ such that $p_{i+1} \in \mathsf{Su}(t_i(p_i))$. Since $\rightsquigarrow$ is an antisymmetric relation, $p_i \neq p_j$ if $i \neq j$.

Let $i_1 = 1$. Suppose that that we have constructed $i_1, \dots, i_k$ such that $i_1 < \cdots < i_k$, $p_{i_k} \in \mathsf{Su}(t_{i_{k-1}} \circ \cdots \circ t_1(p_1))$ and $p_j \notin \mathsf{Su}(t_{i_{k-1}} \circ \cdots \circ t_1(p_1))$ for all $j > i_k$. Set $X_k = \{i > i_k : p_i \in \mathsf{Su}(t_{i_k} \circ \cdots \circ t_{i_1}(p_1))\}$. By the inductive hypothesis, there is $x \in kQ$ and $\lambda \in k^\times$ such that $t_{i_{k-1}} \circ \cdots \circ t_{i_1}(p_1) = \lambda p_{i_k} + x$ with $p_{i_k} \notin \mathsf{Su}(x)$. Since we also know that $p_{i_k+1} \in \mathsf{Su}(t_{i_k}(p_{i_k}))$, and $p_{i_k+1} \notin \mathsf{Su}(t_{i_{k-1}} \circ \cdots \circ t_{i_1}(p_1))$ it follows that $p_{i_k+1} \in \mathsf{Su}(t_{i_k}(p_{i_k}) + x)$. Also, $t_{i_k} \circ \cdots \circ t_{i_1}(p_1) = \lambda t_{i_k}(p_{i_k}) + t_{i_k}(x) = \lambda t_{i_k}(p_{i_k}) + x$, so $p_{i_k+1} \in \mathsf{Su}(t_{i_k} \circ \cdots \circ t_{i_1}(p_1))$. Therefore X_k is not empty. We may define $i_{k+1} = \max X_k$, because X_k is a finite set.

This procedure constructs inductively a strictly increasing sequence of indices $(i_k)_{k \in \mathbb{N}}$ with $p_{i_k} \in \mathsf{Su}(t_{i_{k-1}} \circ \cdots \circ t_{i_1}(p_1))$ for all $k \in \mathbb{N}$. The set $\{t_{i_{k-1}} \circ \cdots \circ t_{i_1}(p_1) : k \in \mathbb{N}\}$ is therefore infinite. This contradicts the reduction-finiteness of $\mathcal{R}$. □

The following characterization of the relation $\rightsquigarrow$ is very useful in practice.

Lemma 1.4. *If p, q are paths, then $q \rightsquigarrow p$ if and only if $p = q$ or there exists a reduction t such that $p \in \mathsf{Su}(t(q))$.*

Proof. First we prove the necessity of the condition. Let $n \in \mathbb{N}$, $r_1, \dots, r_n$ and $p_1, \dots, p_n$ be as in the definition of $\rightsquigarrow$, and suppose that n is minimal. Let $\tilde{p}_1 = p_1$ and for each $i = 1, \dots, n-1$ put $\tilde{p}_{i+1} = r_i(\tilde{p}_i)$. Notice that the minimality implies that $r_i(p_i) \neq p_i$. Let us first show that

$$\text{if } i > j \text{ then } p_i \notin \mathsf{Su}(\tilde{p}_j). \tag{1.1}$$

Suppose otherwise and let (i, j) be a counterexample with j minimal. We will prove that in this situation, $p_l \in \mathsf{Su}(\tilde{p}_l)$ for all $l < j$. We proceed by induction on l. By definition, $p_1 \in \mathsf{Su}(\tilde{p}_1)$. Suppose $1 \leq l < j-1$ and $p_l \in \mathsf{Su}(\tilde{p}_l)$. Then we have $p_{l+1} \in \mathsf{Su}(r_l(p_l))$ and, since $l < j$, $p_{l+1} \notin \mathsf{Su}(\tilde{p}_l)$. Write $\tilde{p}_l = \lambda p_l + x$ with $x \in kQ$ and $p_l \notin \mathsf{Su}(x)$. Since r_l acts nontrivially on p_l, it acts trivially on x; it follows that $r_l(\tilde{p}_l) = \lambda r_l(p_l) + x$ and so $p_{l+1} \in \mathsf{Su}(r_l(\tilde{p}_l)) = \mathsf{Su}(\tilde{p}_{l+1})$. In particular $p_{j-1} \in \mathsf{Su}(\tilde{p}_{j-1})$. Since $p_i \notin \mathsf{Su}(\tilde{p}_{j-1})$ and $p_i \in \mathsf{Su}(\tilde{p}_j)$, we must have $p_i \in \mathsf{Su}(r_{j-1}(p_{j-1}))$.

Now, let $m = n + j - i$, $t_k = r_k$ and $u_k = p_k$ if $k \leq j-1$, and $t_k = r_{i+k-j}$ and $u_k = p_{i+k-j}$ if $j \leq k \leq m$. Then, $u_1 = q$, $u_{n+j-i} = p$ and $u_{k+1} \in \mathsf{Su}(t_k(u_k))$ for all $k = 1, \dots, m-1$. Since $m < n$ this contradicts the choice of n. We thus conclude that (1.1) holds.

We can use the same inductive argument as before to prove that $p_i \in \mathsf{Su}(\tilde{p}_i)$ for all $1 \leq i \leq n$. Denoting $t = (r_n, \dots, r_1)$, observe that $p \in \mathsf{Su}(t(q))$.

Let us now prove the converse. Suppose p and q are distinct paths such that there exists a reduction t with $p \in \mathsf{Su}(t(q))$. Let $t = (t_m, \dots, t_1)$ with t_i basic reductions for all i, and let us proceed by induction on m. Notice that if $m = 1$ there is nothing to prove. Suppose $m > 1$. Let $t_1(q) = \sum_{i=1}^{n} \lambda_i c_i$, with c_i paths and $\lambda_i \in k^\times$ for all $i = 1, \dots, n$. In particular, $q \rightsquigarrow c_i$ for all i. Denote $t' = (t_m, \dots, t_2)$. Since $p \in \mathsf{Su}(t'(t_1(q))) = \mathsf{Su}(\sum_{i=1}^{n} \lambda_i t'(c_i))$, we deduce that there exists $1 \leq i_0 \leq n$ such that $p \in \mathsf{Su}(t'(c_{i_0}))$. By inductive hypothesis we deduce $c_{i_0} \rightsquigarrow p$. We have already seen that $q \rightsquigarrow c_{i_0}$. Therefore $q \rightsquigarrow p$. ☐

Finally, we define a relation $\preceq$ on the set $k^\times Q_{\geq 0} := \{\lambda p : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$ as the least reflexive and transitive relation such that $\lambda p \preceq \mu q$ whenever there exists a reduction r such that $r(\mu q) = \lambda p + x$ with $p \notin \mathsf{Su}(x)$. We state $0 \preceq \lambda p$ for all $\lambda p \in k^\times Q_{\geq 0}$.

If $x = \sum_{i=1}^{n} \lambda_i p_i \in kQ$ with $\lambda_i \in k^\times$ and λp belongs to $k^\times Q_{\geq 0}$, we write $x \preceq \lambda p$ if $\lambda_i p_i \preceq \lambda p$ for all i. If in addition $x \neq \lambda p$ we also write $x \prec p$.

Lemma 1.5. *The binary relation $\preceq$ is an order satisfying the descending chain condition and it is compatible with concatenation.*

Proof. In order to prove the first claim, let us first prove that if $p \in Q_{\geq 0}$ is such that there exists a reduction r with $r(p) = \lambda p + x$ and $p \notin \mathsf{Su}(x)$, then $\lambda = 1$ and $x = 0$. Suppose not. For r a basic reduction, this has already been done in Lemma 1.3. If r is not basic, then $r = (r_n, \dots, r_1)$ with r_i basic and $n \geq 2$. Let $r' = (r_n, \dots, r_2)$. Since $p \in \mathsf{Su}(r(p)) = \mathsf{Su}(r'(r_1(p)))$, there exists $p_1 \in \mathsf{Su}(r_1(p))$ such that $p \in \mathsf{Su}(r'(p_1))$. By the previous case, we obtain that $p \notin \mathsf{Su}(r_1(p))$, so $p \neq p_1$. As a consequence of Lemma 1.4, we know that $p \rightsquigarrow p_1$ since $p_1 \in \mathsf{Su}(r_1(p))$ and that $p_1 \rightsquigarrow p$ since $p \in \mathsf{Su}(r'(p_1))$. This contradicts the antisymmetry of $\rightsquigarrow$.

It is an immediate consequence of the previous fact that given a path p and a reduction t,

$$\text{if } t(\lambda_1 p) = \lambda_2 p + x \text{ with } p \notin \mathsf{Su}(x)\text{, then } \lambda_1 = \lambda_2. \tag{1.2}$$

Let $\lambda_1, \dots, \lambda_{n+1} \in k^\times$, $p_1, \dots, p_{n+1} \in Q_{\geq 0}$, $x_1, \dots, x_n \in kQ$ and reductions $t_1, \dots, t_n$ be such that $t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i$, $p_{i+1} \notin \mathsf{Su}(x_i)$ and $\lambda_{n+1} p_{n+1} = \lambda_1 p_1$. This implies that $p_i \rightsquigarrow p_{i+1}$ for each $1 \leq i \leq n$ and $p_{n+1} = p_1$. Since $\rightsquigarrow$ is antisymmetric, it follows that $p_i = p_1$ for all i and (1.2) implies that $\lambda_i = \lambda_1$ for all i. We thus see that $\preceq$ is antisymmetric.

Let now $(\lambda_i p_i)_{i \in \mathbb{N}}$ be a sequence in $k^\times Q_{\geq 0}$ and $(t_i)_{i \in \mathbb{N}}$ a sequence of reductions such that $t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i$ with $p_{i+1} \notin \mathsf{Su}(x_i)$. Then $p_i \rightsquigarrow p_{i+1}$ for all i and since $\rightsquigarrow$ satisfies the descending chain condition there exists i_0 such that $p_i = p_{i_0}$ for all $i \geq i_0$. Observation (1.2) implies then that $\lambda_i = \lambda_{i_0}$ for all $i \geq i_0$, so that the sequence $(\lambda_i p_i)_{i \in \mathbb{N}}$ stabilizes.

The second claim follows from the following observation. If $r_{a',\rho,c'}$ is a basic reduction with $\rho = (s, f)$ and a, c are paths such that $aa'sc'c \neq 0$, then $a(r_{a',\rho,c'}(q))c = r_{aa',\rho,c'c}(aqc)$ for all $q \in Q_{\geq 0}$. This proves that if $r = (r_m, \dots, r_1)$ with $r_i = r_{a'_i,\rho_i,c'_i}$ a basic reduction for all i, then $a(r(q))c = \tilde{r}(aqc)$ where $\tilde{r} = (\tilde{r}_n, \dots, \tilde{r}_1)$ and $\tilde{r}_i = r_{aa'_i,\rho_i,c'_ic}$ for all i. □

1.2 The Diamond Condition

Let I be a two-sided ideal in kQ.

Definition 1.6. We say that a reduction system $\mathcal{R}$ satisfies *the Diamond condition* for I if

1. the ideal I is equal to the two sided ideal generated by the set $\{s - f\}_{(s,f) \in \mathcal{R}}$,
2. every path is reduction-unique and
3. for each $(s, f) \in \mathcal{R}$, f is irreducible.

One of the main reasons why these reduction systems are useful is the following lemma, which is part of Bergman's Diamond Lemma.

Lemma 1.7. *If the reduction system $\mathcal{R}$ satisfies the Diamond condition for I, then the set $\mathcal{B}$ of irreducible paths satisfies the following properties,*

(i) *$\mathcal{B}$ is closed under divisors,*

(ii) *$\pi(b) \neq \pi(b')$ for all $b, b' \in \mathcal{B}$ with $b \neq b'$,*

(iii) *$\{\pi(b) : b \in \mathcal{B}\}$ is a basis of A.*

Definition 1.8. If $\mathcal{R}$ is a reduction system satisfying the Diamond condition for I, we define $S_{\mathcal{R}} := \{s \in Q_{\geq 0} : (s, f) \in \mathcal{R}, \text{ for some } f \in kQ\}$. If it is not ambiguous, we will write S instead of $S_{\mathcal{R}}$.

Remark 1.9. Given $q \in Q_{\geq 0}$, q is irreducible if and only if there exists no $p \in S$ such that p divides q.

Remark 1.10. In view of Lemma 1.7, we can define a k-linear map $i : A \longrightarrow kQ$ such that be $i(\pi(b))) = b$ for all $b \in \mathcal{B}$. We denote by $\beta : kQ \longrightarrow kQ$ the composition $i \circ \pi$. Notice that if p is a path and r is a reduction such that $r(p)$ is irreducible, then $r(p) = \beta(p)$. In the bibliography, $\beta(p)$ is sometimes called the normal form of p.

Corollary 1.11. *Given a path p, its normal form $\beta(p)$ is such that $\beta(p) \preceq p$. Moreover, $\beta(p) \prec p$ if and only if $p \notin \mathcal{B}$.*

Proof. There is a reduction r such that $\beta(p) = r(p) = \sum_{i=1}^{n} \lambda_i p_i$. We have $\lambda_i p_i \preceq p$ for all i and so $\beta(p) \preceq p$. The last claim follows from the fact that $\beta(p) = p$ if and only if $p \in \mathcal{B}$. □

Next we will prove that these kind of reduction system exists for any two-sided ideal I.

Proposition 1.12. *If $I \subseteq kQ$ is a two-sided ideal, then there exists a reduction system $\mathcal{R}$ which satisfies the Diamond condition for I.*

We will prove this result by putting together a series of lemmas.

Let $\leq$ be a well-order on the set $Q_0 \cup Q_1$. Let $\omega : Q_1 \longrightarrow \mathbb{N}$ be a function and extend it to $Q_{\geq 0}$ defining $\omega(e) = 0$ for all $e \in Q_0$ and $\omega(c_n \cdots c_1) = \sum_{i=1}^{n} \omega(c_i)$ if $c_i \in Q_1$ and $c_n \cdots c_1$ is a path. Given $c, d \in Q_{\geq 0}$ we write that $c \leq_\omega d$ if

- $\omega(c) < \omega(d)$, or
- $c, d \in Q_0$ and $c \leq d$, or
- $\omega(c) = \omega(d)$, $c = c_n \cdots c_1$, $d = d_m \cdots d_1 \in Q_{\geq 1}$ and there exists $j \leq \min(|c|, |d|)$ such that $c_i = d_i$ for all $\in \{1, \ldots, j-1\}$ and $c_j < d_j$.

Notice that the order $\leq_\omega$ is in fact the *deglex* order with weight ω, and it has the following two properties:

1. If $p, q \in Q_{\geq 0}$ and $p \leq_\omega q$, then $cpd \leq_\omega cqd$ for all $c, d \in Q_{\geq 0}$ such that $cpd \neq 0$ and $cqd \neq 0$ in kQ.
2. For all $q \in Q_{\geq 0}$ the set $\{p \in Q_{\geq 0} : p \leq_\omega q\}$ is finite.

It is straightforward to prove the first claim. For the second one, let $\{c^i\}_{i\in\mathbb{N}}$ be a sequence in $Q_{\geq 0}$ such that $c^{i+1} \leq_\omega c^i$ for all i. If $c^i \in Q_0$ for some i, then it is evident that the sequence stabilizes, so let us suppose that $\{c^i\}_{i\in\mathbb{N}}$ is contained in $Q_{\geq 1}$ and $c^{i+1} <_\omega c^i$ for all $i \in \mathbb{N}$. Since $(\omega(c^i))_{i\in\mathbb{N}}$ is a decreasing sequence of natural numbers, it must stabilize, so we may also suppose that $\omega(c^i) = \omega(c^j)$ for all i, j and that the lengths of the paths are bounded above by some $M \in \mathbb{N}$. By definition of $\leq_\omega$, we know that the sequence of first arrows of elements of $\{c^i\}_{i\in\mathbb{N}}$ forms a decreasing sequence in $(Q_1, \leq)$, which must stabilize because $(Q_1, \leq)$ is well-ordered. Let $N \in \mathbb{N}$ be such that the first arrow of c^i equals the first arrow of c^j for all $i, j \geq N$. If $c^i = c^i_{n_i} \cdots c^i_1$, and we denote $c'^i = c^i_{n_i} \cdots c^i_2$, then $\{c'^i\}_{i\geq N}$ is a decreasing sequence in $(Q_{\geq 0}, \leq_\omega)$ with $|c'^i| = M - 1$ for all i. Iterating this process we arrive to a contradiction.

Definition 1.13. Consider as before a well-order $\leq$ on $Q_0 \cup Q_1$ and $\omega : Q_1 \longrightarrow \mathbb{N}$, and $\leq_\omega$ be constructed from them. If $p \in kQ$ and $p = \sum_{i=1}^n \lambda_i c_i$ with $\lambda_i \in k^\times$, $c_i \in Q_{\geq 0}$ and $c_i <_\omega c_1$ for all $i \neq 1$, we write $\mathsf{tip}(p)$ for c_1. If $X \subseteq kQ$, we let $\mathsf{tip}(X) := \{\mathsf{tip}(x) : x \in X \setminus \{0\}\}$.

Consider the set

$$S := \mathsf{Mintip}(I) = \{p \in \mathsf{tip}(I) : p' \notin \mathsf{tip}(I) \text{ for all proper divisors } p' \text{ of } p\}.$$

Notice that if s and s' both belong to S and $s \neq s'$, then s does not divide s'. For each $s \in S$, choose $f_s \in kQ$ such that $s - f_s \in I$, $f_s <_\omega s$ and f_s is parallel to s.

Describing the set $\mathsf{tip}(I)$ is not easy in general. We comment on this problem in the next section.

Lemma 1.14. *Let $\leq_\omega$ and S be as before. The ideal I equals the two sided ideal generated by the set $\{s - f_s\}_{s\in S}$, which we will denote by $\langle s - f_s\rangle_{s\in S}$.*

Proof. It is clear that $\langle s - f_s\rangle_{s\in S}$ is contained in I. Choose $x = \sum_{i=1}^n \lambda_i c_i \in I$ with $\lambda_i \in k^\times$ and $c_i \in Q_{\geq 0}$. We may suppose that $c_1 = \mathsf{tip}(x)$, so that $c_1 \in \mathsf{tip}(I)$. There is a divisor s of c_1 such that $s \in \mathsf{tip}(I)$ and $s' \notin \mathsf{tip}(I)$ for all proper divisor s' of s and $s \in S$ by definition of S. Let $a, c \in Q_{\geq 0}$ with $asc = c_1$.

Define $x' := af_sc + \sum_{i=2}^n \lambda_i c_i$. We have $x = \lambda_1 c_1 + \sum_{i=2}^n \lambda_i c_i = \lambda_1 a(s - f_s)c + x'$, so that $x' \in I$ and, by property (1) of the order $\leq_\omega$, we see that $c_1 > \mathsf{tip}(x')$. We can apply this procedure again to x' and iterate: the process will stop by property (2) and we conclude that $x \in \langle s - f_s\rangle_{s\in S}$. □

Lemma 1.15. *Let $\leq_\omega$ and S be as before. The set $\mathcal{R} := \{(s, f_s)\}_{s\in S}$ is a reduction system such that every path is reduction-unique.*

Proof. Since $s >_\omega \operatorname{tip}(f_s)$ for all $s \in S$, properties (1) and (2) guarantee that every path is reduction-finite. We need to prove that every path is reduction-unique. Recall that π is the canonical projection $kQ \longrightarrow kQ/I$. Let p be a path. Since $I = \langle s - f_s\rangle_{s\in S}$, we see that $\pi(r(p)) = \pi(p)$ for any reduction r. Let r and t be reductions such that $r(p)$ and $t(p)$ are both irreducible. Then, $\pi(r(p) - t(p)) = \pi(p) - \pi(p) = 0$, so that $r(p) - t(p) \in I$. If this difference is not zero, then the path $d = \operatorname{tip}(r(p) - t(p))$ can be written as $d = asc$ with a, c paths and $s \in S$. It follows that the reduction $r_{a,s,c}$ acts nontrivially either on $r(p)$ or on $t(p)$, and this is a contradiction. □

This lemma implies that for each $s \in S$, there exists a reduction r and an irreducible element f'_s such that $r(f_s) = f'_s$. Consider the reduction system $\mathcal{R}' := \{(s, f'_s) : s \in S\}$. The set of irreducible paths for $\mathcal{R}$ coincides with the set of irreducible paths for $\mathcal{R}'$ and, since $\pi(s - f'_s) = \pi(s - f_s) = 0$, we have that $\langle s - f'_s\rangle_{s\in S} \subseteq I$. From Bergman's Diamond Lemma it follows that $I = \langle s - f'_s\rangle_{s\in S}$. The fact that each f'_s is obtained from f_s by applying reductions implies that every reduction relative to $\mathcal{R}'$ is a composition of reductions relative to $\mathcal{R}$, and this implies that every path is reduction unique relative to $\mathcal{R}$. Also, the paths f'_s are irreducible relative to $\mathcal{R}$, and since $S_{\mathcal{R}} = S_{\mathcal{R}'}$, we deduce that the elements f'_s are irreducible relative to $\mathcal{R}'$. We can conclude that the reduction system $\mathcal{R}'$ satisfies the Diamond condition, thereby proving Proposition 1.12.

Lemma 1.16. *Let $\leq_\omega$ and S as before. For each $s \in S$, let f_s and g_s be elements in kQ such that $s - f_s, s - g_s \in I$. If f'_s and g'_s are the elements constructed as in the last paragraph, then $f'_s = g'_s$.*

Proof. Both f'_s and g'_s are irreducible and so their difference $f'_s - g'_s$ is irreducible. On the other hand, $s - f'_s, s - g'_s \in I$ implies that $f'_s - g'_s \in I$. We deduce that $f'_s - g'_s = 0$. □

This lemma implies that the reduction system $\mathcal{R}'$ associated to the order $\leq$ on $Q_0 \cup Q_1$ and weight ω does not depend on any choice.

Definition 1.17. The reduction system $\mathcal{R}'$ associated to the order $\leq$ and weight ω will be denoted by $\mathcal{R}_{\leq,\omega}$.

Notice that if instead of starting from a well-order we consider a total order on $Q_0 \cup Q_1$ and a weight function ω such that $\leq_\omega$ is a well-order on $Q_{\geq 0}$, the construction of a reduction system for I can be carried out in the same way as before.

It is important to emphasize that different choices of orders on $Q_0 \cup Q_1$ and of weights ω will give very different reduction systems, some of which will better suit our purposes than others. Moreover, there are reduction systems which cannot be obtained by this procedure.

1.3 Finding reduction systems that satisfy the Diamond condition

Given an algebra $A = kQ/I$, we proved in Lemmas 1.14 and 1.15 that it is always possible to construct a reduction system $\mathcal{R}$ which satisfies the Diamond condition. However, it is not always easy to follow the prescriptions given by these lemmas for a concrete algebra.

Remark 1.18. The two sided ideal I is usually presented giving a set $X \subseteq kQ$ of generating relations, with X at most countable. From these relations it is easy to obtain a reduction system $\mathcal{R}$ verifying the following conditions:

1. the ideal I is equal to the two sided ideal generated by the set $\{s - f\}_{(s,f)\in\mathcal{R}}$,

2′. every path is reduction-finite,

To do this fix a well-order on $Q_0 \cup Q_1$, a function $\omega : Q_1 \longrightarrow \mathbb{N}$ and consider the total order $\leq_\omega$ on $Q_{\geq 0}$. For all $x \in X$ we can write in a unique way $x = s_x - f_x$, and we can eventually rescale x so that s_x is monic, with $s_x >_\omega f_x$. Define $\mathcal{R}_X := \{(s_x, f_x)\}_{x\in X}$. The set $\mathcal{R}_X$ is a reduction system that satisfies conditions 1 and 2′.

Remark 1.19. Suppose we have a reduction system $\mathcal{R}$ satisfying conditions 1 and 2′ for some ideal I. From $\mathcal{R}$ we can construct another reduction system $\mathcal{R}'$ satisfying conditions 1, 2′ and also

3. for each $(s, f) \in \mathcal{R}$, f is irreducible.

Indeed, since every path is reduction finite, for each $(s, f) \in \mathcal{R}$ we can find a reduction $r_{(s,f)}$ such that $r_{(s,f)}(f)$ is irreducible. The reduction system $\mathcal{R}' := \{(s, r_{(s,f)}(f)) : (s, f) \in \mathcal{R}\}$ verifies conditions 2′ and 3. Let us see that it also verifies condition 1 for I. Notice that $\pi(r_{(s,f)}(f)) = \pi(f) = \pi(s)$, and then $\pi(s - r_{(s,f)}(f)) = 0$. This implies $\langle s - r_{(s,f)}(f)\rangle_{(s,f)\in\mathcal{R}} \subseteq \langle s - f\rangle_{(s,f)\in\mathcal{R}} = I$. In order to see the other inclusion, write $\mathcal{R} = \{(s_i, f_i)\}_{i\in\mathbb{N}}$ such that $s_i \leq s_{i+1}$ for all $i \in \mathbb{N}$. We shall proceed by induction on i and prove that $s_i - f_i \in \langle s - r_{(s,f)}(f)\rangle_{(s,f)\in\mathcal{R}}$ for all $i \in \mathbb{N}$. The element f_1 is irreducible, since every term appearing in it is strictly smaller than s_1 and so it is strictly smaller than all s_i with $i \in \mathbb{N}$. Thus, $r_{(s_1,f_1)}(f_1) = f_1$. Let $i > 1$ and suppose $r_j - f_j \in \langle s - r_{(s,f)}(f)\rangle_{(s,f)\in\mathcal{R}}$ for all $j \leq i$. Notice that the element f_{i+1} can only have terms divisible by s_j with $j \leq i$. In particular, we have that $r_{(s_{i+1},f_{i+1})} = (r_n, \cdots, r_1)$ with $r_k = r_{a_k,(s_{j_k},f_{j_k}),c_k}$ for $j_k \leq i$ and some paths a_k, c_k. The next lemma shows that $r_{(s_{i+1},f_{i+1})}(f_{i+1}) - f_{i+1} \in \langle s_j - f_j\rangle_{j=1}^{i}$. By inductive hypothesis, this implies $r_{(s_{i+1},f_{i+1})}(f_{i+1}) - f_{i+1} \in \langle s - r_{(s,f)}(f)\rangle_{(s,f)\in\mathcal{R}}$, and so, $s_{i+1} - f_{i+1} \in \langle s - r_{(s,f)}(f)\rangle_{(s,f)\in\mathcal{R}}$.

Lemma 1.20. *Let* $r = (r_n, \cdots, r_1)$ *be a reduction with* $r_k = r_{a_k,(s_k,f_k),c_k}$ *basic reductions for* $k = 1, \ldots, n$. *If* $x \in kQ$, *then* $x - r(x)$ *belongs to the two-sided ideal generated by the set* $\{s_k - f_k\}_{k=1}^{n}$.

Proof. We will proceed by induction on n. If $n = 1$, then $r = r_1 = r_{a,(s,f),c}$ is a basic reduction. Suppose $r(x) \neq x$, otherwise there is nothing to prove. We have $x = \lambda asc + x'$, with $\lambda \in k^\times$ and $asc \notin \mathsf{Su}(x')$. Therefore $r(x) = \lambda afc + x'$, and we obtain $x = \lambda a(s-f)c + r(x)$. If $n > 1$, then $x - r(x) = x - r_1(x) + r_1(x) - r'(r_1(x))$, where $r' = (r_n, \dots, r_2)$, and the result follows. □

Observe that different choices of reductions $r_{(s,f)}$ may lead to different reduction systems $\mathcal{R}'$. Also, notice that $S_{\mathcal{R}'} = S_{\mathcal{R}}$.

We have proved the following lemma.

Lemma 1.21. *Let $\leq$ be an order on $Q_0 \cup Q_1$ and ω a weight function. If X is a, at most countable, generating set of relations for I, then $\mathcal{R}_X$ satisfies conditions 1 and $2'$. If $\mathcal{R}_X$ does not satisfy condition 3, using the method given in Remark 1.19 we can find another generating set of relations X' such that $\mathcal{R}_{X'}$ is a reduction system that satisfies conditions $1, 2'$ and 3.*

Proof. The proof was carried in remarks 1.18 and 1.19. Using that notation, we have $X' := \{s - f : (s,f) \in \mathcal{R}'\}$. □

Example 1.22. Consider the quiver Q with only one vertex and four arrows a, b, c and d. Define the order $a < b < c < d$ and $\omega(\alpha) = 1$ for all $\alpha \in Q_1$.

Let $X = \{cd - ab, ab - d\}$ and let I be the two-sided ideal generated by X. The reduction system $\mathcal{R}_X$ is $\{(cd, ab), (ab, d)\}$ and it does not satisfy condition 3 since the path ab is reducible. The reduction $r_{1,ab,1}$ satisfies that $r_{1,ab,1}(ab) = d$ is irreducible. We replace the element (cd, ab) in $\mathcal{R}_{X'}$ by the element (cd, d). Observe that $cd - d = cd - ab + ab - d \in I$, and so the ideal generated by X is the same as the ideal generated by $X' = \{cd - d, ab - d\}$. The reduction system $\mathcal{R}' = \{(cd, d), (ab, d)\}$ satisfies conditions 1, $2'$ and 3.

The next example shows that there are reduction systems satisfying conditions $1, 2'$ and 3 that cannot be obtained by the above method.

Example 1.23. Consider the algebra

$$A = k\langle x, y, z\rangle / (x^3 + y^3 + z^3 - xyz)$$

and let $\mathcal{R} = \{(xyz, x^3 + y^3 + z^3)\}$. It is clear that $\mathcal{R}$ satisfies conditions 1 and 3. Also, but not entirely evident, this reduction system satisfies condition $2'$.

On the other hand, if $\leq$ is an order on the set of arrows $\{x, y, z\}$ and ω is a weight function, the maximum of the set $\{xyz, x^3, y^3, z^3\}$ with respect to the total order $\leq_\omega$ is u^3, where u is the maximum of the set $\{x, y, z\}$ with respect to $\leq$. This implies that there is no choice of order $\leq$ and weight ω such that $xyz = s$, $f = x^3 + y^3 + z^3$ and $f <_\omega s$. Therefore, the reduction system $\mathcal{R}$ cannot be obtained by the method given in remarks 1.18 and 1.19.

Using Bergman's Diamond Lemma with the partial order $\rightsquigarrow$ of Lemma 1.3 we obtain an easy way to check whether a reduction system $\mathcal{R}$ which verifies $1, 2'$ and 3, satisfies the Diamond condition or not.

Proposition 1.24. *Let $\mathcal{R}$ be a reduction system such that conditions 1, 2′ and 3 are satisfied. The reduction system $\mathcal{R}$ satisfies the Diamond condition if and only if all ambiguities are resolvable. In fact, it is enough to check that all minimal overlap ambiguities are resolvable.*

Example 1.25.

1. Consider the algebra $A = k\langle x, y\rangle/I$, where $I = \langle x^2, y^2, yx - \xi xy\rangle$. Let $y < x$ and $\omega(x) = 1 = \omega(y)$. Define $X = \{x^2, y^2, yx - \xi xy\}$. Since $yx >_\omega xy$ we see that the reduction system constructed as in Remark 1.18 is $\mathcal{R}_X = \{(x^2, 0), (y^2, 0), (yx, \xi xy)\}$ which satisfies conditions $1, 2'$ and 3. Denote $\rho_1 = (x^2, 0)$, $\rho_2 = (yx, \xi xy)$ and $\rho_3 = (y^2, 0)$.

   It is clear that there are no inclusion ambiguities relative to $\mathcal{R}_X$ and that the set of minimal overlap ambiguities is

   $$\{(\rho_1, \rho_1, x, x, x), (\rho_2, \rho_1, y, x, x), (\rho_3, \rho_2, y, y, x), (\rho_3, \rho_3, y, y, y)\}.$$

   The ambiguity $(\rho_1, \rho_1, x, x, x)$ is resolvable since we can choose $r = r_{1,\rho_1,x}$ and $t = r_{x,\rho_1,1}$, and we obtain $r(x^3) = 0 = t(x^3)$. Similarly, the ambiguity $(\rho_3, \rho_3, y, y, y)$ is resolvable. Let us see that the ambiguity $(\rho_3, \rho_2, y, y, x)$ is resolvable. Consider $r = r_{1,\rho_3,x}$ and $t = (r_{x,\rho_3,1}, r_{1,\rho_2,y}, r_{y,\rho_2,1})$. Then $r(p) = 0 = t(p)$. The process of reducing p from both sides can pictured as follows:

$$y^2x = (yy)x \xrightarrow{r} (0)x,$$
$$y^2x = y(yx) \xrightarrow{t_1} y(\xi xy) = \xi yxy = \xi(yx)y \xrightarrow{t_2} \xi(\xi xy)y \longrightarrow \xi^2 xyy = \xi^2 x(yy) \xrightarrow{t_3} 0,$$

   where $t = (t_3, t_2, t_1)$. Similarly, to see that yx^2 is resolvable, consider $r = (r_{1,\rho_1,y}, r_{x,\rho_2,1}, r_{1,\rho_2,x})$ and $t = r_{y,\rho_1,1}$. Therefore, the reduction system $\mathcal{R}_X$ satisfies the Diamond condition for I.

2. The reduction system $\mathcal{R}$ of Example 1.23 does not have any ambiguities and therefore it satisfies the Diamond condition.

   On the other hand, consider the order $x < y < z$ and weight function ω with $\omega(x) = \omega(y) = \omega(z) = 1$. Define $X = \{x^3 + y^3 + z^3 - xyz\}$. The reduction system $\mathcal{R}_X$ is $\{(z^3, xyz - x^3 - y^3)\}$ and it satisfies conditions $1, 2'$ and 3. Denote $\rho = (z^3, xyz - x^3 - y^3)$. There are no inclusion ambiguities and the only minimal overlap ambiguity is (ρ, ρ, z, z^2, z), but it is not resolvable. Indeed, $r_{z,\rho,1}(z^4) = zxyz - zx^3 - zy^3$ which is irreducible. On the other hand, $r_{1,\rho,z}(z^4) = xyz^2 - x^3z - y^3z$ which is also irreducible and different from the first one. Therefore (ρ, ρ, z, z^2, z) is not resolvable and we obtain that $\mathcal{R}_X$ does not satisfy the Diamond condition.

Remark 1.26. We now deal with the case in which a reduction system constructed from a generating set X for I, an order $\leq$ and a weight function ω, as in remarks 1.18 and 1.19, satisfies conditions $1, 2'$ and 3 but not the Diamond condition. An example of such a reduction system is $\mathcal{R}_X$ of the last example. This procedure is also described in [3], Section 5.

Suppose $\mathcal{R}_X$ satisfies conditions $1, 2'$ and 3 but there exists a non resolvable ambiguity $p = (\rho_1, \rho_2, a, b, c)$. Choose two reductions $r = (r_n, \dots, r_1)$ and $t = (t_m, \dots, t_1)$ such that $r(p)$ and $t(p)$ are irreducible but different, with r_1 and t_1 the basic reductions corresponding to the type ambiguity. Notice that $r(p) - t(p) \in I \setminus \{0\}$.

Write $r(p) - t(p) = s - f$ with $f <_\omega s$ and define $X' := X \cup \{s - f\}$. The reduction system $\mathcal{R}_{X'}$ satisfies conditions $1, 2'$ and we may have to modify it as in Remark 1.19 so that it satisfies condition 3. Notice that p is now resolvable with respect to $\mathcal{R}_{X'}$. New ambiguities may now appear, so it is necessary to iterate this process, which may have infinitely many steps, but we will arrive to a reduction system $\mathcal{R}$ that satisfies conditions $1, 2', 3$, and having no non resolvable ambiguities, that is, $\mathcal{R}$ will satisfy the Diamond condition.

Next we give an example to illustrate this procedure.

Example 1.27. Consider the algebra A, the set X and the reduction system $\mathcal{R}_X = \{(z^3, xyz - x^3 - y^3)\}$ of the second example of Example 1.25. We saw that (ρ, ρ, z, z^2, z) is a non resolvable ambiguity and that the elements $r(z^4) = zxyz - zx^3 - zy^3$, $t(z^4) = xyz^2 - x^3z - y^3z$ are irreducible, where $r = r_{z,\rho,1}$ and $t = r_{1,\rho,z}$. The difference between $r(p)$ and $t(p)$ is $xyz^2 - x^3z - y^3z - zxyz + zx^3 + zy^3$. Define

$$X_1 = X \cup \{xyz^2 - x^3z + y^3z + zxyz - zx^3 - zy^3\}.$$

We have $\mathcal{R}_{X_1} = \{\rho, \rho_1\}$ where

$$\rho_1 = (xyz^2, x^3z + y^3z + zxyz - zx^3 - zy^3),$$

and notice that the ambiguity (ρ, ρ, z, z^3, z) is now resolvable. The set of ambiguities for $\mathcal{R}_{X_1}$ is $\{(\rho, \rho, z, z^2, z), (\rho_1, \rho, xy, z^2, z)\}$. Applying reductions to the element xyz^3 we obtain again two different irreducible elements whose difference is

$$y^3z^2 + x^3z^2 + z^2xyz - z^2x^3 - z^2y^3 - xyxyz + xyx^3 + xy^4.$$

Define

$$X_2 = X_1 \cup \{y^3z^2 + x^3z^2 + z^2xyz - z^2x^3 - z^2y^3 - xyxyz + xyx^3 + xy^4\}.$$

The reduction system $\mathcal{R}_{X_3}$ is the set $\{\rho, \rho_1, \rho_2\}$, where

$$\rho_3 = (y^3z^2, -x^3z^2 - z^2xyz + z^2x^3 + z^2y^3 + xyxyz - xyx^3 - xy^4).$$

We obtain the new ambiguity $(\rho_3, \rho, y^3, z^2, z)$ which is not difficult to see that it is resolvable. Thus, the reduction system

$$\begin{aligned}\mathcal{R}_{X_3} = \{&(z^3, xyz - x^3 - y^3), (xyz^2, x^3z + y^3z + zxyz - zx^3 - zy^3),\\ &(y^3z^2, -x^3z^2 - z^2xyz + z^2x^3 + z^2y^3 + xyxyz - xyx^3 - xy^4)\}.\end{aligned}$$

satisfies the Diamond condition for $I = \langle X \rangle$.

There is another reduction system which verifies the Diamond condition for I, namely

$$\mathcal{R} = \{(xyz, x^3 + y^3 + z^3)\}.$$

This example shows that a reduction system obtained from a $\mathtt{deglex}$ order $\leq_\omega$ may be sometimes less convenient than other ones.

We end this section with a series of results that are important for the next section.

If $\mathcal{R}$ is a reduction system, denote

$$\mathsf{Inc}(\mathcal{R}) := \{abc \in Q_{\geq 0} : (\rho_1, \rho_2, a, b, c) \text{ is an inclusion ambiguity for some } \rho_1, \rho_2 \in \mathcal{R}\}.$$

Proposition 1.28. *Let $\mathcal{R}$ be a reduction system which satisfies the Diamond condition for an ideal* I. *The set $\mathcal{R}' = \{(s, f) \in \mathcal{R} : s \notin \mathsf{Inc}(\mathcal{R})\}$ is a reduction system satisfying the Diamond condition for* I *and it has no inclusion ambiguities.*

Proof. Observe that $\mathcal{R}'$ is obtained from $\mathcal{R}$ by deleting the elements $(s, f) \in \mathcal{R}$ for which there exists $(s', f') \in \mathcal{R}$ with s' a proper divisor of s. In particular, $\mathcal{R}' \subseteq \mathcal{R}$. Let $J \subseteq I$ be the ideal generated by the set $\{s - f\}_{(s,f) \in \mathcal{R}'}$.

A path p is irreducible relative to $\mathcal{R}'$ if and only if it is irreducible relative to $\mathcal{R}$. Since $\mathcal{R}$ satisfies the Diamond condition for I, we obtain that $\mathcal{R}'$ satisfies the Diamond condition for J. The inclusion $J \subseteq I$ induces a k-algebra epimorphism $\varphi : kQ/J \longrightarrow kQ/I$ and since the set of irreducible paths is the same for both reduction systems, the homomorphism sends a basis of kQ/J to a basis of kQ/I, and so φ is an isomorphism. This implies that $I = J$ and we deduce that $\mathcal{R}'$ satisfies the Diamond condition for I. The fact that $\mathcal{R}'$ does not have any inclusion ambiguity is evident. □

Example 1.29. Consider the quiver Q with only one vertex and four arrows, namely $Q_1 = \{a, b, c, d\}$. Define $a < b < c < d$ and $\omega(\alpha) = 1$ for all $\alpha \in Q_1$. Let X be the set $\{abc - a, abc - b, ab - d\}$ and define I as the two-sided ideal generated by it. We have $\mathcal{R}_X = \{(abc, a), (abc, b), (ab, d)\}$.

Following the methods described in remarks 1.18, 1.19 and 1.26, we obtain the reduction system $\mathcal{R} = \{(abc, a), (abc, b), (ab, d), (dc, a), (b, a), (aa, d)\}$ that satisfies the Diamond condition. The set $\mathsf{Inc}(\mathcal{R})$ is $\{abc, ab\}$. Then, $\mathcal{R}' = \{(dc, a), (b, a), (aa, d)\}$ is a reduction system satisfying the Diamond condition and it has no inclusion ambiguities.

The following corollary says that this method is in fact a very efficient way to compute the reduction systems $\mathcal{R}_{\leq,\omega}$ constructed in Lemmas 1.14 and 1.15, as well as the set $\mathsf{tip}(I)$.

Corollary 1.30. *Let* $\leq$ *be an order on* $Q_0 \cup Q_1$, ω *a weight function and* I *a two-sided ideal. If* X *is a generating set of relations for* I *such that* $\mathcal{R}_X$ *satisfies conditions* $1, 2', 3$ *and every ambiguity is resolvable for it, then*

$$\mathcal{R}_{\leq,\omega} = \{(s,f) \in \mathcal{R}_X : s \notin \mathsf{Inc}(\mathcal{R}_X)\},$$

and $p \in \mathsf{tip}(I)$ *if and only if there exists* $q \in \mathsf{tip}(X)$ *such that* q *divides* p.

Proof. Denote $\mathcal{R}' = \{(s,f) \in \mathcal{R}_X : s \notin \mathsf{Inc}(\mathcal{R}_X)\}$. By the previous proposition, we know that $\mathcal{R}'$ satisfies the Diamond condition for I. Recall that

$$S_{\mathcal{R}_{\leq,\omega}} := \mathsf{Mintip}(I) = \{p \in \mathsf{tip}(I) : p' \notin \mathsf{tip}(I) \text{ for all proper divisors } p' \text{ of } p\}.$$

Let $\mathcal{B}_{\mathcal{R}_{\leq,\omega}}$ and $\mathcal{B}_{\mathcal{R}'}$ be respectively the sets of irreducible paths of $\mathcal{R}_{\leq,\omega}$ and $\mathcal{R}'$. Notice that a path p belongs to $\mathcal{B}_{\mathcal{R}_{\leq,\omega}}$ if and only if it is not divisible by any element of $\mathsf{tip}(I)$.

By the definition of $\mathcal{R}_X$ we have that $f <_\omega s$ for all $(s,f) \in \mathcal{R}'$ and $s - f \in I$, thus $s \in \mathsf{tip}(I)$ and we can find an element $(s', f') \in \mathcal{R}_{\leq,\omega}$ with s' a divisor of s. We deduce that $\mathcal{B}_{\mathcal{R}_{\leq,\omega}} \subseteq \mathcal{B}_{\mathcal{R}'}$. By Lemma 1.7, the sets $\pi(\mathcal{B}_{\mathcal{R}_{\leq,\omega}})$ and $\pi(\mathcal{B}_{\mathcal{R}'})$ are bases of kQ/I and $\pi(\mathcal{B}_{\mathcal{R}_{\leq,\omega}}) \subseteq \pi(\mathcal{B}_{\mathcal{R}'})$, therefore $\pi(\mathcal{B}_{\mathcal{R}_{\leq,\omega}}) = \pi(\mathcal{B}_{\mathcal{R}'})$. Again by Lemma 1.7 we obtain that $\mathcal{B}_{\mathcal{R}_{\leq,\omega}} = \mathcal{B}_{\mathcal{R}'}$.

Let $(s,f) \in \mathcal{R}'$. We know that $\mathcal{R}'$ does not have any inclusion ambiguities. This implies that every proper divisor of s belongs to $\mathcal{B}_{\mathcal{R}'}$. Since $\mathcal{B}_{\mathcal{R}_{\leq,\omega}} = \mathcal{B}_{\mathcal{R}'}$, we deduce that every proper divisor of s does not belong to $\mathsf{tip}(I)$ and therefore $s \in S_{\mathcal{R}_{\leq,\omega}}$. Since f is irreducible relative to $\mathcal{R}'$, and therefore it is irreducible relative to $\mathcal{R}_{\leq,\omega}$, we obtain that $(s,f) \in \mathcal{R}_{\leq,\omega}$. This implies that $\mathcal{R}' \subseteq \mathcal{R}_{\leq,\omega}$.

Let us see the other inclusion. Let $(s,f) \in \mathcal{R}_{\leq,\omega}$. The fact that s is reducible relative to $\mathcal{R}_{\leq,\omega}$ implies that it is reducible relative to $\mathcal{R}'$ and so there exists $(s', f') \in \mathcal{R}'$ with s' a divisor of s. Since every proper divisor of s is irreducible relative to $\mathcal{R}_{\leq,\omega}$, we obtain that every proper divisor of s is irreducible relative to $\mathcal{R}'$ and therefore $s' = s$. From this we deduce that $f - f'$ belongs to I. The element $f - f'$ is irreducible and so $f - f' = 0$, that is, $(s,f) = (s', f') \in \mathcal{R}'$.

Let us see the second claim. Notice that a path p is divisible by an element of $S_{\mathcal{R}'}$ if and only if it is divisible by an element in $\mathsf{tip}(X)$. On the other hand, a path p belongs to $\mathsf{tip}(I)$ if and only if it is divisible by some element in $S_{\mathcal{R}_{\leq,\omega}} = S_{\mathcal{R}'}$. Putting these two facts together we obtain the second claim. ☐

A reduction system $\mathcal{R}$ satisfying the Diamond condition for an ideal I may have elements $(s,f) \in \mathcal{R}$ with $s \in Q_0 \cup Q_1$. The following lemma says how to delete this type of elements.

Lemma 1.31. *If $\mathcal{R}$ is a reduction system satisfying the Diamond condition for* I *with no inclusion ambiguities, then the algebra* $A = kQ/I$ *can be written as* $A = k\hat{Q}/\hat{I}$ *where* $\hat{Q}$ *is a subquiver of* Q*,* $\hat{\mathcal{R}} = \mathcal{R} \setminus \{(s, f) \in \mathcal{R} : s \in Q_0 \cup Q_1\}$ *is a reduction system in* $k\hat{Q}$ *that satisfies the Diamond condition for* $\hat{I}$*, it has no inclusion ambiguities and the set* $\hat{S}$ *of first coordinates of elements of* $\hat{\mathcal{R}}$ *satisfies* $\hat{S} \subseteq \hat{Q}_{\geq 2}$*.*

Proof. Define $\mathcal{R}_i := \{(s, f) \in \mathcal{R} : s \in Q_i\}$ and $S_i := S \cap Q_i$ for $i = 0, 1$. We know that for all $(s, f) \in \mathcal{R}$, f is a linear combination of paths that are not divisible by any element of $S_0 \cup S_1$. Also, if $(s, f) \in \mathcal{R} \setminus (\mathcal{R}_0 \cup \mathcal{R}_1)$, then s is not divisible by any element of $S_0 \cup S_1$. Since every path parallel to a vertex e is divisible by e, we obtain that $f = 0$ for all $(s, f) \in \mathcal{R}_0$. Let X be the set of arrows in Q_1 that have as target or source an element of S_0. Consider the quiver $\hat{Q}$ obtained from Q by removing all the vertices in S_0 as well as all the arrows in $X \cup S_1$. Let $\hat{\mathcal{R}} := \mathcal{R} \setminus \{\mathcal{R}_0 \cup \mathcal{R}_1\}$. Observe that $\hat{\mathcal{R}}$ is a reduction system defined on $\hat{Q}$. Define $\hat{I}$ as the two-sided ideal in $k\hat{Q}$ generated by the set $\{s - f\}_{(s,f)\in\hat{R}}$. Notice that $\hat{I}$ is a subset of I.

A path p in Q is irreducible with respect to $\mathcal{R}$ if and only if it is a path in $\hat{Q}$ that is irreducible with respect to $\hat{\mathcal{R}}$. Also, if p is a path in $\hat{Q}$ and r is a basic reduction of $\mathcal{R}$ such that $r(p) \neq p$, then r is a basic reduction of $\hat{\mathcal{R}}$. These two facts imply that every path in $\hat{Q}$ is reduction-unique with respect to $\hat{\mathcal{R}}$. Thus, $\hat{\mathcal{R}}$ is a reduction system satisfying the Diamond condition for $\hat{I}$. The inclusions $\hat{Q} \subseteq Q$ and $\hat{I} \subseteq I$ induce a k-algebra morphism $\varphi : k\hat{Q}/\hat{I} \longrightarrow kQ/I$. The set of paths in Q that are irreducible with respect to $\mathcal{R}$ coincides with the set of paths in $\hat{Q}$ that are irreducible with respect to $\hat{\mathcal{R}}$. Since these sets form bases of kQ/I and $k\hat{Q}/\hat{I}$ respectively, we see that φ is an isomorphism. □

Example 1.32. Consider the reduction system $\mathcal{R}' = \{(dc, a), (b, a), (aa, d)\}$ of Example 1.29. Using the notation of the above proof we have $\mathcal{R}_0 = \emptyset, S_0 = \emptyset, \mathcal{R}_1 = \{(b, a)\}$ and $S_1 = \{b\}$. Therefore $\hat{Q}_0 = Q_0, \hat{Q}_1 = \{a, c, d\}, \hat{\mathcal{R}} = \{(dc, a), (aa, d)\}, \hat{I} = \langle dc - a, aa - d\}$ and we have an isomorphism of k-algebras

$$kQ/\langle abc - a, abc - b, ab - d\rangle \cong k\hat{Q}/\langle dc - a, aa - d\},$$

where $\hat{\mathcal{R}}$ is a reducion system satisfying the Diamond condition for $\hat{I}$ with no inclusion ambiguities and $\hat{S} \subseteq \hat{Q}_2$.

1.4 Ambiguities

Let I be a two-sided ideal of kQ and $\mathcal{R}$ a reduction system satisfying the Diamond condition for I. By Proposition 1.28 and Lemma 1.31 we can assume without loss of generality that $\mathcal{R}$ does not have any inclusion ambiguities and that $S \subseteq Q_{\geq 2}$.

We will next recall the definition of n-ambiguity or n-chain according to the terminology used in [1], [2], [32] and to Bardzell's [4] *associated sequences of paths*.

Definition 1.33. Given $n \geq 2$ and $p \in Q_{\geq 0}$,

1. the path p is a *left* n*-ambiguity* if there exist $u_0 \in Q_1$, $u_1, \dots, u_n$ irreducible paths such that
   (a) $p = u_0 u_1 \cdots u_n$,
   (b) for all i, the path $u_i u_{i+1}$ is reducible but $u_i d$ is irreducible for any proper left divisor d of u_{i+1}.
2. The path p is a *right* n*-ambiguity* if there exist $v_0 \in Q_1$ and $v_1, \dots, v_n$ irreducible paths such that
   (a) $p = v_n \cdots v_0$,
   (b) for all i, the path $v_{i+1} v_i$ is reducible but $d v_i$ is irreducible for any proper right divisor of v_{i+1}.

Define $\mathcal{A}_{-1} := Q_0, \mathcal{A}_0 := Q_1, \mathcal{A}_1 := S$ and for $n \geq 2$ define $\mathcal{A}_n$ and $\mathcal{A}'_n$ to be respectively the set of left n-ambiguities and right n-ambiguities.

Proposition 1.34. *For all* $n \geq 2$ *the equality* $\mathcal{A}_n = \mathcal{A}'_n$ *holds.*

Proof. This fact is proved in [4] and also in [32]. □

Proposition 1.35. *Let* $n, m \in \mathbb{N}$, $p \in Q_{\geq 1}$. *If* $u_0, \hat{u}_0 \in Q_1$ *and* $u_1, \dots u_n, \hat{u}_1, \dots, \hat{u}_n$ *are paths in* Q *such that both* $u_0, \dots, u_n$ *and* $\hat{u}_0, \dots, \hat{u}_n$ *satisfy conditions (1a) and (1b) of the previous definition for* p, *then* $n = m$ *and* $u_i = \hat{u}_i$ *for all* i, $0 \leq i \leq n$.

Proof. Suppose $n \leq m$. It is obvious that $u_0 = \hat{u}_0$, since both of them are arrows. Notice that $kQ = T_{kQ_0} kQ_1$, that is the free algebra generated by kQ_1 over kQ_0, which implies that either $u_0 u_1$ divides $\hat{u}_0 \hat{u}_1$ or $\hat{u}_0 \hat{u}_1$ divides $u_0 u_1$, and moreover $u_0 u_1, \hat{u}_0 \hat{u}_1 \in \mathcal{A}_1 = S$. Remark 1.9 says that $u_0 u_1 = \hat{u}_0 \hat{u}_1$. Since $u_0 = \hat{u}_0$, we must have $u_1 = \hat{u}_1$. By induction on i, let us suppose that $u_j = \hat{u}_j$ for $j \leq i$. As a consequence, $u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m$.

If $i + 1 = n$, this reads $u_n = \hat{u}_n \cdots \hat{u}_m$, and the fact that u_n is irreducible and $\hat{u}_j \hat{u}_{j+1}$ is reducible for all $j < m$ implies that $m = n$ and $u_n = \hat{u}_n$. Instead, suppose that $i + 1 < n$. From the equality $u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m$ we deduce that there exists a path d such that $u_{i+1} = \hat{u}_{i+1} d$ or $\hat{u}_{i+1} = u_{i+1} d$. If $u_{i+1} = \hat{u}_{i+1} d$ and $d \in Q_{\geq 1}$, we can write $d = d_2 d_1$ with $d_1 \in Q_1$. The path $\hat{u}_{i+1} d_2$ is a proper left divisor of u_{i+1} and by condition (1b) we obtain that $u_i \hat{u}_{i+1} d_2$ is irreducible. This is absurd since $u_i \hat{u}_{i+1} d_2 = \hat{u}_i \hat{u}_{i+1} d_2$ by inductive hypothesis, and the right hand term is reducible by condition (1b). It follows that $d \in Q_0$ and then $u_{i+1} = \hat{u}_{i+1}$. The case where $\hat{u}_{i+1} = u_{i+1} d$ is analogous. □

Corollary 1.36. *Given* $n, m \geq -1$, $\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ *if* n *and* m *are different.*

Proposition 1.37. *If $n \geq 0$ is even and $p \in \mathcal{A}_n$, then there are exactly two elements $p_1, p_2 \in \mathcal{A}_{n-1}$ dividing p. Moreover, if $p = u_0 \cdots u_n = v_n \dots v_0$ are the factorizations of p as left and right n-ambiguity, then $p_1 = u_0 \dots u_{n-1}$ and $p_2 = v_{n-1} \dots v_0$.*

Proof. This fact is also proved in [4] and in [32] □

Proposition 1.38. *Let* Amb *be the set of ambiguities of $\mathcal{R}$. The map $f : \mathsf{Amb} \longrightarrow \mathcal{A}_2$ given by $f((\rho_1, \rho_2, a, b, c)) = abc$ is well defined and it is a bijection.*

Proof. The fact that $\mathcal{R}$ has no inclusion ambiguities implies that all the ambiguities are minimal overlap ambiguities. Let $(\rho_1, \rho_2, a, b, c)$ be an ambiguity and write $a = a_0 a_1$ with $a_0 \in Q_1$. Define $u_0 := a_0$, $u_1 := a_1 b$ and $u_2 = c$. Since $(\rho_1, \rho_2, a, b, c)$ is a minimal overlap ambiguity and there are no inclusion ambiguities, if $abc = d_1 s d_2$ with $s \in S$, then either $d_1 \in Q_0$ and $s = ab$, or $d_2 \in Q_0$ and $s = bc$. For any proper left divisor u' of u_1, the path $u_0 u'$ is a proper left divisor of ab and therefore it is irreducible. Similarly, for any proper left divisor u' of u_2, the path $u_1 u'$ is a proper left divisor of $u_1 u_2 = a_1 bc$ and we deduce that it is irreducible. This implies that $u_0 u_1 u_2 = abc$ is a 2-ambiguity, and so the map f is well defined.

Let $u_0 u_1 u_2 \in \mathcal{A}_2$. We know that $u_0 u_1$ is reducible. Since $u_0 \in Q_1$ and u_1 is irreducible we obtain that $u_0 u_1 \in S$. On the other hand, $u_1 u_2$ is reducible and since u_2 and $u_1 u'$ are irreducible for all proper left divisor u' of u_2, we can write $u_1 = u_1' u_1''$ such that $u_1'' u_2 \in S$. The element $(\rho_1, \rho_2, u_0 u_1', u_1'', u_2)$ is a minimal overlap ambiguity, where ρ_1, ρ_2 are respectively the unique elements in $\mathcal{R}$ with first coordinate $u_0 u_1$ and $u_1'' u_2$. Notice that $f((\rho_1, \rho_2, u_0 u_1', u_1'', u_2)) = u_0 u_1 u_2$ and so f is suryective.

Let us see that f is injective. Let $(\rho_1, \rho_2, a, b, c)$ and $(\rho_1', \rho_2', a', b', c')$ be minimal overlap ambiguities such that $abc \neq a'b'c'$. If $abc = a'b'c'$, then either ab divides $a'b'$ or $a'b'$ divides ab. Since $ab, a'b' \in S$ and there are no inclusion ambiguities, we obtain that $ab = a'b'$ and so $c = c'$. Similarly, $bc = b'c'$ and we deduce that $b = b'$ and $a = a'$. This implies that $(\rho_1, \rho_2, a, b, c) = (\rho_1', \rho_2', a', b', c')$. □

Example 1.39. In practice, computing the sets $\mathcal{A}_n$ is not difficult. Proposition 1.38 is the best way to start since the computation of the set Amb is not hard. To obtain the set $\mathcal{A}_3$ we have to consider the *minimal overlaps* of elements $p = u_0 u_1 u_2 \in \mathcal{A}_2$ with elements $s \in S$ on the right in such a way that s does not overlap with $u_0 u_1$. In general to obtain $\mathcal{A}_{n+1}$ from $\mathcal{A}_n$ we form every *minimal overlap* of elements $p = u_0 \dots u_n$ with elements $s \in S$ on the right such that s does not overlap with $u_{n-1} u_n$.

Next we give an example. Let Q be the quiver with one vertex and two arrows a, b. Let I be the two-sided ideal generated by the set $X = \{aba, bab\}$ and consider the order $a < b$ and weight $\omega(a) = \omega(b) = 1$. By Proposition 1.24, the reduction system $\mathcal{R}_X = \{(aba, 0), (bab, 0)\}$ satisfies the Diamond condition for I. Moreover, it has no inclusion ambiguities and $S = \{aba, bab\} \subseteq Q_{\geq 2}$.

Using Proposition 1.38 we deduce that $\mathcal{A}_2 = \{abab, baba\}$. Let $u_0 = a, u_1 = ba, u_2 = b$, and $u'_0 = b, u'_1 = ab, u'_2 = a$. We obtain $abab = u_0u_1u_2$ and $baba = u'_0u'_1u'_2$. Notice that $abab$ overlaps on the right with both aba and bab, but aba overlaps also with u_0u_1. The element $baba$ also overlaps with aba and bab, but bab overlaps with $u'_0u'_1$. Therefore $\mathcal{A}_3 = \{ababab, bababa\}$. If we define $u_3 = ab$ and $u'_3 = ba$ we can write $ababab = u_0u_1u_2u_3$ and $bababa = u'_0u'_1u'_2u'_3$.

Continuing in this way we obtain $\mathcal{A}_4 = \{abababa, bababab\}$. We deduce that $\mathcal{A}_n = \{(ab)^{\frac{n}{2}+1}, (ba)^{\frac{n}{2}+1}\}$ for n even, and $\mathcal{A}_n = \{(ab)^{\frac{n+1}{2}}a, (ba)^{\frac{n+1}{2}}b\}$ for n odd.

We end this section with a proposition that indicates how to compute ambiguities for a particular family of algebras.

Proposition 1.40. *Suppose* $S \subseteq Q_2$. *For all* $n \geq 1$,

$$\mathcal{A}_n = \{\alpha_0 \dots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1 \textit{ for all } i \textit{ and } \alpha_{i-1}\alpha_i \in S\}$$

Moreover, given $p = \alpha_0 \dots \alpha_n \in \mathcal{A}_n$, *we can write* p *as a left ambiguity choosing* $u_i = \alpha_i$, *for all* i, *and as a right ambiguity choosing* $v_i = \alpha_{n-i}$.

Proof. We proceed by induction on n. If $n = 1$ we know that $\mathcal{A}_1 = S$ in which case there is nothing to prove. Let $u_0 \cdots u_n u_{n+1} \in \mathcal{A}_{n+1}$ and suppose that the result holds for all $p \in \mathcal{A}_n$. Since $u_0 \cdots u_n$ belongs to $\mathcal{A}_n$ we only have to prove that $u_{n+1} \in Q_1$ and that $u_n u_{n+1} \in S$. We know that $u_n \in Q_1$, that u_{n+1} is irreducible and that $u_n u_{n+1}$ is reducible. As a consequence, there exist $s \in S$ and $\nu \in Q_{\geq 0}$ such that $u_n u_{n+1} = s\nu$. Moreover, $u_n d$ is irreducible for any proper left divisor d of u_{n+1}, so the only possibility is $\nu \in Q_0$. We conclude that $u_n u_{n+1}$ belongs to S. Since $S \subseteq Q_2$ and $u_n \in Q_1$, we deduce that $u_{n+1} \in Q_1$. This proves that $\mathcal{A}_{n+1} \subseteq \{\alpha_0 \cdots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1$ for all i and $\alpha_{i-1}\alpha_i \in S\}$. The other inclusion is clear. □

Example 1.41. Let $\xi \in k$ and Q be the quiver with one vertex and two arrows. Denote $Q_1 = \{x, y\}$. Consider $A = k\langle x, y\rangle / I$ with $I = \langle x^2, y^2, yx - \xi xy\rangle$. In Examples 1.25 we proved that the set $\mathcal{R} = \{(x^2, 0), (y^2, 0), (yx, \xi xy)\}$ is a reduction system satisfing the Diamond condition. Also, there are no inclusion ambiguities and $S = \{x^2, y^2, yx\} \subseteq Q_2$. The only path of length two not in S is xy. By Proposition 1.40, the set $\mathcal{A}_n$ coincides with the set of paths not divisible by xy, that is

$$\mathcal{A}_n = \{y^s x^t : s + t = n + 1\}.$$

Chapter 2

Previous work by Anick, Green and Bardzell

In this chapter we will recall the main results in [1], [2], [4] and [32]. Bardzell's resolution and Sköldberg's contracting homotopy for it are fundamental for our results.

2.1 Anick's resolution

Let A be an augmented k-algebra, that is a k-algebra with a one dimensional module T. Let $\{v\}$ be a k-basis of T and $\epsilon : A \longrightarrow T$ the A-module morphism given by $\epsilon(a) = a \cdot v$, for $a \in A$, and let $\eta : T \longrightarrow A$ be the k-module morphism given by $\eta(\lambda v) = \lambda \cdot 1$, for $\lambda \in k$.

The algebra A can be presented as $A = k\langle X \rangle / I$ for some set X and some ideal I. Choose for example a generating set X of A. Consider the quiver Q with one vertex and $Q_1 = X$. Let $\leq$ be a total order on X and $\omega : X \longrightarrow \mathbb{N}$ a function such that $\leq_\omega$, the induced order $\leq_\omega$ on the set of paths, is a well-order. For example, if $\leq$ is a well-order on X, then $\leq_\omega$ is a well-order. Consider the reduction system $\mathcal{R}_{\leq,\omega}$ coming from this order. Let $\mathcal{A}_n$ be the set of n-ambiguities for $n \geq -1$ and let $\mathcal{B}$ be the set of irreducible paths.

For $n \geq 0$ define a partial order on the set $\{p \otimes b : p \in \mathcal{A}_n, b \in \mathcal{B}\} \subseteq k\mathcal{A}_n \otimes_k A$ by $p \otimes b < p' \otimes b'$ if $pb <_\omega p'b'$. This is in fact a total order. Observe that the set $\{p \otimes b : p \in \mathcal{A}_n, b \in \mathcal{B}\}$ is a basis of $k\mathcal{A}_n \otimes_k A$. For $x \in k\mathcal{A}_n \otimes_k A$ denote $\mathrm{tip}(x)$ to be the highest term when writing x as a linear combination of elements of this basis.

Theorem 2.1 (Anick [1])**.** *There is a resolution of* T *by free right* A*-modules,*

$$0 \longleftarrow T \xleftarrow{\epsilon} A \xleftarrow{d_0} kX \otimes_k A \xleftarrow{d_1} kS \otimes_k A \xleftarrow{d_2} k\mathcal{A}_2 \otimes_k A \xleftarrow{d_3} \cdots, \qquad (2.1)$$

where $d_0(x) = x - \eta(\epsilon(x))$ *for* $x \in X$, *and for each* $n \geq 1$ *the differential* d_n *is such that*

$$\mathsf{tip}(d_n(u_0 \cdots u_n \otimes 1) - u_0 \cdots u_{n-1} \otimes u_n) < u_0 \cdots u_{n-1} \otimes u_n. \tag{2.2}$$

The first problem with this result is that the information about the differentials is very limited. For an ambiguity $u_0 \cdots u_n$, there might be many terms $p \otimes b$ verifying $p \otimes b < u_0 \otimes u_{n-1} \otimes u_n$. The second problem is that it is an existence theorem, even if we are able to control these terms and we manage to obtain morphisms $\{d_n\}_{n \in \mathbb{N}}$ forming a complex and verifying condition 2.2, we cannot conclude from Anick's theorem that it will be a resolution.

We give an example to illustrate these problems.

Example 2.2. Let $\xi \in k$ and $A = k\langle x, y\rangle / \left(x^2, y^2, yx - \xi xy\right)$. Consider the order $x < y$ and weights $\omega(x) = 1 = \omega(y)$. By Example 1.25 $\mathcal{R} = \{(x^2, 0), (y^2, 0), (yx, \xi xy)\}$ satisfies the Diamond condition for I. Also, Example 1.41 shows that the set of n-ambiguities is $\mathcal{A}_n = \{y^s x^t : s + t = n + 1\}$. For $n \in \mathbb{N}$, the set $\{y^s x^t \otimes u : s + t = n + 1, u \in \{1, x, y, xy\}\}$ is a basis of $k\mathcal{A}_n \otimes_k A$.

Let T be the one-dimensional module with basis $\{v\}$ and action $v \cdot x = 0 = v \cdot y$. By Anick's theorem we know that there exists a resolution as in 2.1 such that $d_0(x) = x$, $d_0(y) = y$ and for $s + t = n + 1$,

$$\mathsf{tip}(d_n(y^s x^t \otimes 1) - y^s x^{t-1} \otimes x) < y^s x^{t-1} \otimes x, \tag{2.3}$$

in case $t > 0$, and

$$\mathsf{tip}(d_n(y^{n+1} \otimes 1) - y^n \otimes y) < y^n \otimes y. \tag{2.4}$$

Suppose $t > 0$. Equation (2.3) says that $d_n(y^s x^t \otimes 1) = y^s x^{t-1} \otimes x + w$, where w is a linear combination of terms of the form $y^{s'} x^{t'} \otimes u$ with $s' + t' = n$ and $y^{s'} x^{t'} u < y^s x^t$. Since $s' + t' = n$ we obtain that u cannot be equal to xy, otherwise the weight of $y^{s'} x^{t'} u$ is $n + 2$ which is higher than the weight of $y^s x^t$. Since $x < y$ we deduce that $u = x$ or $u = 1$. Moreover, any term of the form $y^{s'} x^{t'} \otimes u$ where $s' + t' = n$, veryfing either $u = x$ and $t' < t$ or $u = 1$, satisfies $y^{s'} x^{t'} \otimes u \leq y^s x^{t-1} \otimes x$. There are $n + 1 + t$ of these terms.

In case $t = 0$ similar arguments show that $d_n(y^{n+1} \otimes 1) = y^n \otimes y + w$, where w is a linear combination of terms of the form $y^{s'} x^{t'} \otimes u$ with $s' + t' = n$ and $u \in \{1, x, y\}$. There are $3n + 3$ of these terms.

In any case, condition 2.2 of Anick's theorem provides very poor information about the differentials.

Let us consider the special case $\xi = 0$. The family of A-module morphisms $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n(y^s x^t \otimes u) = y^s x^{t-1} \otimes xu$, for $t > 0$, and $f_n(y^{n+1} \otimes u) = y^n \otimes yu$, form a complex and verifies

$$\mathsf{tip}(f_n(u_0 \ldots u_n \otimes 1) - u_0 \ldots u_{n-1} \otimes u_n) < u_0 \ldots u_{n-1} \otimes u_n,$$

but so far there is no guarantee that $(k\mathcal{A}_\bullet \otimes_k A, f_\bullet)$ is a resolution of T. We will be able to prove that it is a resolution using Bardzell's resolution for monomial algebras.

2.2 Anick-Green resolution

In [2], the authors generalize the results in [1] to the setting of quotients of path algebras by ideals contained in J^2, where J is the ideal generated by the arrows. Let $\pi : kQ \longrightarrow A$ be the canonical projection, where $A = kQ/I$.

Consider a reduction system $\mathcal{R}_{\leq,\omega}$ coming from a well order $\leq$ on the set of $Q_0 \cup Q_1$ and a weight function ω. For $n \geq -1$, let $\mathcal{A}_n$ be the sets of n-ambiguities and $\mathcal{B}$ be let the set of irreducible paths. Given $1 \leq i \leq m$, denote $\mathcal{A}_n^i = \{p \in \mathcal{A}_n : e_i p = p\}$. For each $b \in \mathcal{B}$ and $n \geq -1$, consider $W_n^b = \langle p \otimes \pi(b') \in k\mathcal{A}_n^i \otimes_E A : p \in \mathcal{A}_n^i, b' \in \mathcal{B}, pb' <_\omega b \rangle_k \subseteq k\mathcal{A}_n^i \otimes_E A$.

Theorem 2.3 (Anick-Green, [2]). *With the above notation, let* T *be the one dimensional module over the vertex* e_i*. There is a resolution of* T *by projective right* A*-modules,*

$$0 \longleftarrow T \overset{\epsilon}{\longleftarrow} A \overset{d_0}{\longleftarrow} k\mathcal{A}_0^i \otimes_E A \overset{d_1}{\longleftarrow} k\mathcal{A}_1^i \otimes_E A \overset{d_2}{\longleftarrow} k\mathcal{A}_2^i \otimes_E A \overset{d_3}{\longleftarrow} \cdots,$$

and for each $n \geq 1$ *the differential* d_n *is such that*

$$d_n(u_0 \dots u_n \otimes 1) - u_0 \dots u_{n-1} \otimes \pi(u_n) \in W_{n-1}^{u_0 \dots u_n}.$$

Each one of the problems to construct resolutions using Anick's theorem appears also in this generalization.

2.3 Bardzell's resolution

Let A be a monomial algebra, that is, a quotient of a path algebra $A = kQ/I$ with I a two-sided ideal generated by paths of length at least 2. Let X be the set

$$X = \{x \in Q_{\geq 0} : x \in I \text{ and } x' \notin I \text{ for all proper divisor } x' \text{ of } x\}.$$

Notice that I is the two-sided ideal generated by X.

Lemma 2.4. *The reduction system* $\mathcal{R} = \{(x, 0) : x \in X\}$ *satisfies the Diamond condition for* I*, it has no inclusion ambiguities and* $S_{\mathcal{R}} = X \subseteq Q_{\geq 2}$*.*

Denote $E = kQ_0$. The fact that the sets of n-ambiguities are subsets of $Q_{\geq 2}$ implies that the k-vector space $k\mathcal{A}_n$ is in fact an E-bimodule and therefore we can form the A-bimodules $A \otimes_E k\mathcal{A}_n \otimes_E A$. The set $\{\pi(b) \otimes p \otimes \pi(b') : p \in \mathcal{A}_n, b, b' \in \mathcal{B}\}$ is a basis of $A \otimes_E k\mathcal{A}_n \otimes_E A$. Let $\mathcal{B}$ be the set of irreducible paths and $\pi : kQ \longrightarrow A$ the canonical projection.

Consider the following sequence of A-bimodules,

$$\cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

$$A \otimes_E A \xrightarrow{\cong} A \otimes_E k\mathcal{A}_{-1} \otimes_E A$$

where

1. $d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ for $n \geq 0$,
2. $d_{-1}(a \otimes b) = ab$ is multiplication in A,
3. if n is even, $q \in \mathcal{A}_n$ and $q = u_0 \cdots u_n = v_n \cdots v_0$ are respectively the factorizations of q as left and right n-ambiguity,
$$d_n(1 \otimes q \otimes 1) = \pi(v_n) \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes \pi(u_n),$$
4. if n is odd and $q \in \mathcal{A}_n$,
$$d_n(1 \otimes q \otimes 1) = \sum_{\substack{apc=q \\ p \in \mathcal{A}_{n-1}, a,c \in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c).$$

In [4], Bardzell proves the following theorem.

Theorem 2.5 (Bardzell, [4]). *The sequence* $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ *is the minimal projective* A*-bimodule resolution of* A.

A contracting homotopy for this resolution is given in [32] by Sköldberg. For $n = -1$, $s_n : A \longrightarrow kQ \otimes_E k\mathcal{A}_{-1} \otimes_E A$ is the $kQ - E$-bimodule map given by $s_{-1}(a) = a \otimes 1$, for $a \in kQ$.

For $n \in \mathbb{N}_0$, $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ is given by

$$s_n(1 \otimes q \otimes \pi(b)) = (-1)^{n+1} \sum_{\substack{apc=qb \\ p \in \mathcal{A}_n, a,c \in Q_{\geq 0}}} \pi(a) \otimes p \otimes \pi(c),$$

with $b \in \mathcal{B}$ and $q \in \mathcal{A}_{n-1}$.

Example 2.6. Consider the algebra A of Example 2.2 for $\xi = 0$, that is, $A = k\langle x, y\rangle / I$ with I the two sided ideal generated by the paths x^2, y^2, yx. In this case $E = k$. We have already proved that $\mathcal{A}_n = \{y^s x^t : s + t = n + 1\}$ and $\mathcal{B} = \{1, x, y, xy\}$.

Given $q \in \mathcal{A}_n$, there are $s, t \in \mathbb{N}$ such that $s + t = n + 1$ and $q = y^s x^t$. Suppose $q = apc$ with $p = y^{s'} x^{t'} \in \mathcal{A}_{n-1}$ and $a, c \in Q_{\geq 0}$. Since $s + t = n + 1$ and $s' + t' = n$, either a belongs to Q_0 and c belongs to Q_1 or $a \in Q_1$ and $c \in Q_0$. Using this fact we obtain that the Bardzell resolution for A is

$$\cdots \xrightarrow{d_2} A \otimes_k k\mathcal{A}_1 \otimes_k A \xrightarrow{d_1} A \otimes_k k\mathcal{A}_0 \otimes_k A \xrightarrow{d_0} A \otimes_k A \xrightarrow{d_{-1}} A \longrightarrow 0,$$

where the differentials are given as follows. Let $n \geq 0$ and $s, t \in \mathbb{N}_0$ such that $s + t = n + 1$, so that $y^s x^t \in \mathcal{A}_n$.

1. If n is even,

$$\begin{aligned}
d_n(1 \otimes y^s x^t \otimes 1) &= y \otimes y^{s-1} x^t \otimes 1 - 1 \otimes y^s x^{t-1} \otimes x, \quad \text{if } t > 0 \text{ and } s > 0. \\
d_n(1 \otimes x^{n+1} \otimes 1) &= x \otimes y^n \otimes 1 - 1 \otimes x^n \otimes x, \\
d_n(1 \otimes y^{n+1} \otimes 1) &= y \otimes y^n \otimes 1 - 1 \otimes y^n \otimes y.
\end{aligned}$$

2. If n is odd,

$$\begin{aligned}
d_n(1 \otimes y^s x^t \otimes 1) &= y \otimes y^{s-1} x^t \otimes 1 + 1 \otimes y^s x^{t-1} \otimes x, \quad \text{if } t > 0 \text{ and } s > 0. \\
d_n(1 \otimes x^{n+1} \otimes 1) &= x \otimes y^n \otimes 1 + 1 \otimes x^n \otimes x, \\
d_n(1 \otimes y^{n+1} \otimes 1) &= y \otimes y^n \otimes 1 + 1 \otimes y^n \otimes y.
\end{aligned}$$

Let T be the one dimensional module of Example 2.2, that is, the one dimensional module with basis $\{v\}$ such that $v \cdot x = 0 = v \cdot y$. Applying the functor $T \otimes_A (-)$ to Bardzell resolution of A we obtain that the sequence $(k\mathcal{A}_\bullet \otimes_k A, f_\bullet)$ described at the end of Example 2.2 is a resolution of T by free right A-modules.

For the study of the general case $A = k\langle x, y\rangle / \left(x^2, y^2, yx - \xi xy\right)$ with $\xi \in k$ see Chapter 4

Chapter 3

Projective resolutions using ambiguities

In this chapter we state and prove our main results, namely Theorem 3.5 and Theorem 3.6. Fix an algebra $A = kQ/I$ and a reduction system $\mathcal{R}$ satisfying the Diamond condition for I with no inclusion ambiguities and such that $S \subseteq Q_{\geq 2}$.

Definition 3.1. There is a monomial algebra associated to A and $\mathcal{R}$, defined as

$$A_S := kQ/\langle S\rangle.$$

Let $\pi' : kQ \longrightarrow A_S$ be the canonical projection. Notice that S is the minimal set of paths generating the monomial ideal $\langle S\rangle$, and since the set $\mathcal{B}$ of irreducible paths coincides with the set of paths not divisible by any element of S, we obtain that the set $\pi'(\mathcal{B})$ is a k-basis of A_S. The algebra A_S is a generalization of the algebra A_{mon} defined in [20].

From the formulas of the differentials of Bardzell resolution of A_S we are going to define morphisms of A-bimodules $\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$. Notice that the kQ- bimodule $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ is a k-vector space with basis $\{a \otimes q \otimes c : a, c \in Q_{\geq 0}, q \in \mathcal{A}_n, aqc \neq 0 \text{ in } kQ\}$. Consider the following sequence,

$$\cdots \underset{s_2}{\overset{f_2}{\rightleftarrows}} kQ \otimes_E k\mathcal{A}_1 \otimes_E kQ \underset{s_1}{\overset{f_1}{\rightleftarrows}} kQ \otimes_E k\mathcal{A}_0 \otimes_E A \underset{s_0}{\overset{f_0}{\rightleftarrows}} kQ \otimes_E kQ \underset{s_{-1}}{\overset{f_{-1}}{\rightleftarrows}} kQ \longrightarrow 0$$

$$kQ \otimes_E kQ \;\Big\downarrow\cong\; kQ \otimes_E k\mathcal{A}_{-1} \otimes_E kQ$$

where

1. $f_{-1}(a \otimes b) = ab$,
2. if n is even, then define f_n as the unique k-linear map such that for all $a, c \in Q_{\geq 0}$, $q \in \mathcal{A}_n$ such that $aqc \neq 0$ in kQ and $q = u_0 \cdots u_n = v_n \cdots v_0$ are respectively

the factorizations of q as left and right n-ambiguity,

$$f_n(a \otimes q \otimes c) = av_n \otimes v_{n-1} \cdots v_0 \otimes c - a \otimes u_0 \cdots u_{n-1} \otimes u_n c,$$

and notice that it is a kQ-bimodule morphism.

3. If n is odd then f_n is the unique k-linear map such that for $a, c \in Q_{\geq 0}, q \in \mathcal{A}_n$ as before,

$$f_n(a \otimes q \otimes c) = \sum_{\substack{a'pc'=q \\ p \in \mathcal{A}_{n-1}, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'c,$$

and it is kQ-bimodule morphism.

4. $S_{-1}(x) = x \otimes 1$ and if $n \geq 0$, S_n is the unique k-linear map such that for all $a, c \in Q_{\geq 0}, q \in \mathcal{A}_{n-1}$,

$$S_n(a \otimes q \otimes c) = (-1)^{n+1} \sum_{\substack{a'pc'=qc \\ p \in \mathcal{A}_n, a', c' \in Q_{\geq 0}}} aa' \otimes p \otimes c'.$$

Notice that S_n is a $kQ - E$-bimodule map for all $n \geq -1$.

As we have already done for A in Remark 1.10, we define a k-linear map $i' : A_S \longrightarrow kQ$ such that $i'(\pi'(b))) = b$ for all $b \in \mathcal{B}$, and we denote by $\beta' : kQ \longrightarrow kQ$ the composition $i' \circ \pi'$. Given $n \geq -1$, let us fix notation for the following k-linear maps:

$$\begin{aligned} \pi_n &:= \pi \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi, & \pi'_n &:= \pi' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes \pi', \\ i_n &:= i \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i, & i'_n &:= i' \otimes \mathrm{id}_{k\mathcal{A}_n} \otimes i', \\ \beta_n &:= i_n \circ \pi_n, & \beta'_n &:= i'_n \circ \pi'_n. \end{aligned}$$

The maps f_n induce, respectively, A-bimodule maps

$$\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$$

where

$$\delta_n := \pi_{n-1} \circ f_n \circ i_n,$$

and A_S-bimodule maps

$$\delta'_n : A_S \otimes_E k\mathcal{A}_n \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S$$

defined by

$$\delta'_n := \pi'_{n-1} \circ f_n \circ i'_n.$$

Remark 3.2. Notice that δ_{-1} and δ'_{-1} are respectively multiplication in A and in A_S. Also, observe that the sequence

$$\cdots \xrightarrow{\delta'_2} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E A_S \xrightarrow{\delta'_{-1}} A_S \longrightarrow 0\ ,$$

is, by definition, the Bardzell resolution of A_S. So the morphisms δ_n with $n \geq -1$ are morphisms defined by *copying* the formulas of Bardzell's differentials of A_S. The morphisms δ_n are not going to satisfy $\delta_{n-1} \circ \delta_n = 0$, but they are going to be a starting point from where we are going to obtain differentials $d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ that form a resolution of A.

The maps S_n induce $A-E$-bimodule maps $s_n : A \otimes_E k\mathcal{A}_{n-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ and $A_S - E$-bimodule maps $s'_n : A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S \longrightarrow A_S \otimes_E k\mathcal{A}_n \otimes_E A_S$ by the formulas

$$s_n := \pi_n \circ S_n \circ i_{n-1},$$
$$s'_n := \pi'_n \circ S_n \circ i'_{n-1}.$$

Observe that $(s'_n)_{\geq -1}$ is the Sköldberg's contracting homotopy for the Bardzell's resolution of A_S.

Recall that in Chapter 1 we constructed, from the reduction system $\mathcal{R}$, a partial order $\preceq$ on the set $k^{\times}Q_{\geq 0}$. We define some sets that will be useful in the sequel. For any $n \geq -1$ and $\mu q \in k^{\times}Q_{\geq 0}$, consider the following subsets of $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$:

- $\mathcal{L}_n^{\preceq}(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda apc \preceq \mu q\}$,
- $\mathcal{L}_n^{\prec}(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda apc \prec \mu q\}$,

and the following subsets of $A \otimes_E k\mathcal{A}_n \otimes_E A$:

- $\overline{\mathcal{L}}_n^{\preceq}(\mu q) := \{\lambda \pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \preceq \mu q\}$,
- $\overline{\mathcal{L}}_n^{\prec}(\mu q) := \{\lambda \pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \prec \mu q\}$.

Remark 3.3. We observe that

$$f_{n+1}(x) \in \langle \mathcal{L}_n^{\preceq}(\mu q) \rangle_{\mathbb{Z}}, \quad \text{for all } x \in \mathcal{L}_{n+1}^{\preceq}(\mu q), \text{ and}$$
$$S_n(x) \in \langle \mathcal{L}_n^{\preceq}(\mu q) \rangle_{\mathbb{Z}}, \quad \text{for all } x \in \mathcal{L}_{n-1}^{\preceq}(\mu q).$$

Remark 3.4. Notice that even if we start from a reduction system of the form $\mathcal{R}_{\leq,\omega}$, the partial order $\preceq$ compares very few thing when compared with the order $\leq_\omega$ used by Anick and Green, and so the sets $\overline{\mathcal{L}}_n^{\prec}$ are considerably smaller than the sets W_n^b used in Anick-Green theorem. The fact that we can use the partial order $\preceq$ is already a big improvement, this will become apparent in the examples of Chapter 4.

We will now state the main theorems. Recall that our aim is to construct, for non necessarily monomial algebras, a bimodule resolution starting from the Bardzell's resolution of the monomial algebra A_S. The first theorem says that if the difference between its differentials and the monomial differentials can be "controlled", then we will actually obtain an exact complex. The second theorem says that it is possible to construct the differentials.

Theorem 3.5. *Set* $d_{-1} := \delta_{-1}$ *and* $d_0 := \delta_0$. *Given* $N \in \mathbb{N}_0$ *and morphisms of* A*-bimodules* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ *for* $1 \leq i \leq N$. *If*

1. $d_{i-1} \circ d_i = 0$ *for all* i, $1 \leq i \leq N$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q) \rangle_k$ *for all* $i \in \{1, \dots, N\}$ *and for all* $q \in \mathcal{A}_i$,

then the complex

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

is exact.

Theorem 3.6. *There exist* A*-bimodule morphisms* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ *for* $i \in \mathbb{N}_0$ *and* $d_{-1} : A \otimes_E A \longrightarrow A$ *such that*

1. $d_{i-1} \circ d_i = 0$, *for all* $i \in \mathbb{N}_0$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q) \rangle_{\mathbb{Z}}$ *for all* $i \geq -1$ *and* $q \in \mathcal{A}_i$.

We will carry out the proofs of these theorems in the following section.

3.1 Proofs of the theorems

We keep the same notations and conditions of the previous section. We start by proving some technical lemmas.

Lemma 3.7. *Given* $n \geq 0$, *the following equalities hold*

1. $\delta_n \circ \pi_n = \pi_{n-1} \circ f_n$,
2. $\delta'_n \circ \pi'_n = \pi'_{n-1} \circ f_n$.

The proof is straightforward after the definitions.

Next we prove three lemmas where we study how the maps defined in the previous section behave with respect to the partial order $\preceq$.

Lemma 3.8. *For all* $n \in \mathbb{N}_0$ *and* $\mu q \in k^\times Q_{\geq 0}$*, the images by* π_n *of* $\mathcal{L}_n^{\preceq}(\mu q)$ *and of* $\mathcal{L}_n^{\prec}(\mu q)$ *are respectively contained in* $\langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$ *and in* $\langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$*.*

Proof. Given $n \in \mathbb{N}_0$, $\mu q \in k^\times Q_{\geq 0}$ and $x = \lambda a \otimes p \otimes c \in \mathcal{L}_n^{\preceq}(\mu q)$, where $a, c \in Q_{\geq 0}$ and $p \in \mathcal{A}_n$, suppose $\beta(a) = \sum_i \lambda_i b_i$ and $\beta(c) = \sum_j \lambda'_j b'_j$. Since $\beta(a) \preceq a$ and $\beta(c) \preceq c$, then $\lambda_i b_i \preceq a$ and $\lambda'_j b'_j \preceq c$ for all i, j. This implies

$$\lambda\lambda_i\lambda_j b_i p b'_j \preceq \lambda a p c \preceq \mu q$$

and so $\lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j)$ belong to $\overline{\mathcal{L}}_n^{\preceq}(\mu q)$ for all i, j. The result follows from the equalities

$$\pi_n(x) = \lambda\pi(a) \otimes p \otimes \pi(c) = \lambda\pi(\beta(a)) \otimes p \otimes \pi(\beta(c)) = \sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j).$$

The proof of the second part is analogous. □

Corollary 3.9. *Let* $n \geq -1$ *and* $\mu q \in k^\times Q_{\geq 0}$*. Keeping the same notations of the proof of the previous lemma, we conclude that*

1. *if* $x \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$*, then* $\lambda\pi(a)x\pi(c) \in \langle \overline{\mathcal{L}}_n^{\preceq}(\lambda\mu a q c)\rangle_{\mathbb{Z}}$*,*
2. *if* $x \in \overline{\mathcal{L}}_n^{\prec}(\mu q)$*, then* $\lambda\pi(a)x\pi(c) \in \langle \overline{\mathcal{L}}_n^{\prec}(\lambda\mu a q c)\rangle_{\mathbb{Z}}$*.*

Lemma 3.10. *Given* $n \in \mathbb{N}_0$ *and* $\mu q \in k^\times Q_{\geq 0}$*, there are inclusions*

1. $\delta_n(\overline{\mathcal{L}}_n^{\preceq}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$*,*
2. $\delta_n(\overline{\mathcal{L}}_n^{\prec}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$*,*
3. $s_n(\overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$*,*
4. $s_n(\overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$*.*

Proof. From $x = \lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$, with $b, b' \in \mathcal{B}$ and $p \in \mathcal{A}_n$, we get $i_n(x) = \lambda b \otimes p \otimes b'$. This element belongs to $\mathcal{L}_n^{\preceq}(\mu q)$ and this implies that $f_n(\lambda b \otimes p \otimes b')$ belongs to $\langle \mathcal{L}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$, by Remark 3.3. As a consequence of Lemma 3.8 we obtain that $\delta_n(x) = \pi_{n-1}(f_n(\lambda b \otimes p \otimes b'))$ belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$. The proofs of the other statements are similar. □

Lemma 3.11. *Given* $n \geq -1$ *and* $\mu q \in k^\times Q_{\geq 0}$*, if* $x = \lambda a \otimes p \otimes c \in \mathcal{L}_n^{\preceq}(\mu q)$ *is such that* $\pi'_n(x) = 0$*, then*

$$\pi_n(x) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}.$$

Proof. By hypothesis we get that $0 = \pi'_n(x) = \pi'(a) \otimes p \otimes \pi'(c)$. The only possibilities are $\pi'(a) = 0$ or $\pi'(c) = 0$, this is, $a \notin \mathcal{B}$ or $c \notin \mathcal{B}$, namely $\beta(a) \prec a$ or $\beta(c) \prec c$.

Writing $\beta(a) = \sum_i \lambda_i b_i$ and $\beta(c) = \sum_j \lambda'_j b'_j$, we deduce that $\lambda\lambda_i\lambda'_j b_i p b_j \prec \mu q$ for all i, j. As a consequence, $\sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

The proof ends by computing

$$\pi_n(x) = \pi_n(\beta(x)) = \pi_n(\sum_{i,j} \lambda\lambda_i\lambda'_j b_i \otimes p \otimes b'_j) = \sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j).$$

□

The importance of the preceding lemmas is that they guarantee how differentials and morphisms used for the homotopy behave with respect to the order. This is stated explicitly in the following corollary.

Corollary 3.12. *Given* $n \geq 1$, $\mu q \in k^\times Q_{\geq 0}$ *and* $x \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$, *the following facts hold:*

1. $\delta_{n-1} \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_{n-2}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$,
2. $x - \delta_{n+1} \circ s_{n+1}(x) - s_n \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

Proof. Let us first write $x = \lambda\pi(b) \otimes p \otimes \pi(b')$ with $b, b' \in \mathcal{B}$ and $x' := i_n(x) = \lambda b \otimes p \otimes b'$. Lemma 3.7 implies that

$$\delta_{n-1} \circ \delta_n(x) = \delta_{n-1} \circ \delta_n \circ \pi_n(x') = \delta_{n-1} \circ \pi_{n-1} \circ f_n(x') = \pi_{n-2} \circ f_{n-1} \circ f_n(x').$$

By Remark 3.3, $f_{n-1} \circ f_n(x') \in \mathcal{L}_{n-2}^{\preceq}(\mu q)$. Next, by Lemma 3.11, in order to prove that $\delta_{n-1} \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_{n-2}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$, it suffices to verify that $\pi'_{n-2} \circ f_{n-1} \circ f_n(x') = 0$, which is in fact true using Lemma 3.7, and the fact that $(A_S \otimes_E k\mathcal{A}_\bullet \otimes_E A_S, \delta'_\bullet)$ is exact.

In order to prove (2), we first remark that if $k \in \mathbb{N}_0$ and $y \in \langle \mathcal{L}_k^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$, then $i'_k \circ \pi'_k(y) - i_k \circ \pi_k(y) \in \langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$. Indeed, let us write $y = \lambda a \otimes p \otimes c \in \mathcal{L}_k^{\preceq}(\mu q)$. In case $a \in \mathcal{B}$ and $c \in \mathcal{B}$, there are equalities $i'_k \circ \pi'_k(y) = y = i_k \circ \pi_k(y)$, and so the difference is zero. If either $a \notin \mathcal{B}$ or $c \notin \mathcal{B}$, then $\pi'_k(y) = 0$ and in this case Lemma 3.11 implies that $\pi_k(y) \in \langle \overline{\mathcal{L}}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$. So, $i_k \circ \pi_k(y) \in \langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$ and the difference we are considering belongs to $\langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

Fix now $x = \lambda\pi(b) \otimes p \otimes \pi(b')$ and $x' = i_n(x) = \lambda b \otimes p \otimes b'$, with $b, b' \in \mathcal{B}$.

Since $x' = i'_n \circ \pi'_n(x')$,

$$\begin{aligned} x - \delta_{n+1} \circ s_{n+1}(x) - s_n \circ \delta_n(x) = \pi_n(x') - \pi_n(f_{n+1} \circ i_{n+1} \circ \pi_{n+1} \circ S_{n+1}(x')) \\ - \pi_n(S_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x')). \end{aligned}$$

The previous comments and Remark 3.3 allow us to write that

$$\pi_n \circ f_{n+1} \circ (i'_{n+1} \circ \pi'_{n+1} - i_{n+1} \circ \pi_{n+1}) \circ S_{n+1}(x') \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}},$$
$$\pi_n \circ S_n \circ (i'_{n-1} \circ \pi'_{n-1} - i_{n-1} \circ \pi_{n-1}) \circ f_n(x') \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}.$$

It is then enough to prove that

$$\pi_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x')) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}},$$

but

$$\begin{aligned} &\pi'_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x')) \\ &= \pi'_n(x') - \delta'_{n+1} \circ s'_{n+1}(\pi'_n(x')) - s'_n \circ \delta'_n(\pi'_n(x')) \\ &= 0. \end{aligned}$$

Finally, we deduce from Lemma 3.11 that

$$\pi_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x')) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}.$$

□

Next we prove another technical lemma that shows how to control the differentials.

Lemma 3.13. *Fix $n \in \mathbb{N}_0$, let R be either k or $\mathbb{Z}$.*

1. *If $d : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ is a morphism of A-bimodules such that $(d - \delta_n)(1 \otimes p \otimes 1) \in \langle \overline{\mathcal{L}}_{n-1}^{\prec}(p)\rangle_R$ for all $p \in \mathcal{A}_n$, then given $x \in \langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_R$, $(d - \delta_n)(x) \in \langle \overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)\rangle_R$ for all $\mu q \in k^{\times}Q_{\geq 0}$.*
2. *If $\rho : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n+1} \otimes_E A$ is a morphism of $A - E$-bimodules such that $(\rho - s_n)(1 \otimes p \otimes \pi(b)) \in \langle \overline{\mathcal{L}}_{n+1}^{\prec}(pb)\rangle_R$, for all $p \in \mathcal{A}_n$ and $b \in \mathcal{B}$, then for all $x \in \langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_R$, $(\rho - s_n)(x)$ belongs to $\langle \overline{\mathcal{L}}_{n+1}^{\prec}(\mu q)\rangle_R$ for all $\mu q \in k^{\times}Q_{\geq 0}$.*

Proof. Given $\mu q \in k^{\times}Q_{\geq 0}$ and $x \in \langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_R$, let us see that $(d - \delta_n)(x) \in \langle \overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)\rangle_R$. It suffices to prove the statement for $x = \lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$.

By hypothesis, $(d - \delta_n)(1 \otimes p \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\prec}(p)\rangle_R$, so $(d - \delta_n)(x)$ equals $\lambda\pi(b)(d - \delta_n)(1 \otimes p \otimes 1)\pi(b')$ and it belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\prec}(\lambda b p b')\rangle_R \subseteq \langle \overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)\rangle_R$, using Corollary 3.9.

The second part is analogous. □

Next proposition will provide the remaining necessary tools for the proofs of Theorem 3.5 and Theorem 3.6.

Proposition 3.14. *Fix* $n \in \mathbb{N}_0$. *Suppose that for each* $i \in \{0, \dots, n\}$ *there are morphisms of* A*-bimodules* $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$, *and morphisms of* $A - E$*-bimodules* $\rho_i : A \otimes_E k\mathcal{A}_{i-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_i \otimes_E A$. *Denote* $d_{-1} = \mu$ *and define* $\rho_{-1} : A \longrightarrow A \otimes_E A$ *as* $\rho(a) = a \otimes 1$.

If the following conditions hold,

(i) $d_{i-1} \circ d_i = 0$ *for all* $i \in \{0, \dots, n\}$,

(ii) $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q) \rangle_R$ *for all* $i \in \{0, \dots, n\}$ *and for all* $q \in \mathcal{A}_i$,

(iii) for all $i \in \{-1, \dots, n-1\}$ *and for all* $x \in A \otimes_E k\mathcal{A}_i \otimes_E A$, $x = d_{i+1} \circ \rho_{i+1}(x) + \rho_i \circ d_i(x)$,

(iv) $(\rho_i - s_i)(1 \otimes q \otimes \pi(b)) \in \langle \overline{\mathcal{L}}^{\prec}_{i}(qb) \rangle_R$ *for all* $i \in \{0, \dots, n\}$, *for all* $q \in \mathcal{A}_i$ *and for all* $b \in \mathcal{B}$,

then:

1. *If* $d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ *is a map satisfying the following conditions:*

   (i) $d_n \circ d_{n+1} = 0$,

   (ii) $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{n}(q) \rangle_R$,

   then there exists a morphism $\rho_{n+1} : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n+1} \otimes_E A$ *of* $A - E$ *bimodules such that*

   (a) for all $x \in A \otimes_E k\mathcal{A}_n \otimes_E A$, $x = d_{n+1} \circ s_{n+1}(x) + s_n \circ d_n(x)$

   (b) for all $q \in \mathcal{A}_n$ *and for all* $b \in \mathcal{B}$, $(\rho_{n+1} - s_{n+1})(1 \otimes q \otimes \pi(b)) \in \langle \mathcal{L}^{\prec}_{n+1}(qb) \rangle_R$.

2. *there exists a morphism of* A*-bimodules* $d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ *such that*

   (i) $d_n \circ d_{n+1} = 0$,

   (ii) $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{n}(q) \rangle_R$.

Proof. In order to prove (2), fix $q \in \mathcal{A}_{n+1}$. By Lemma 3.10, $\delta_{n+1}(1 \otimes q \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}^{\preceq}_{n}(q) \rangle_{\mathbb{Z}}$ and using Lemma 3.13, $(d_n - \delta_n)(\delta_{n+1}(1 \otimes q \otimes 1))$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n-1}(q) \rangle_R$. Corollary 3.12 tells us that $\delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$ is in $\langle \overline{\mathcal{L}}^{\prec}_{n-1}(q) \rangle_{\mathbb{Z}}$. We deduce from the equality

$$d_n(\delta_{n+1}(1 \otimes q \otimes 1)) = \delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1) + (d_n - \delta_n)(\delta_{n+1}(1 \otimes q \otimes 1))$$

that $d_n(\delta_{n+1}(1 \otimes q \otimes 1))$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n-1}(q) \rangle_R$.

Let us define $\tilde{d}_{n+1} : A \times k\mathcal{A}_{n+1} \times A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A$ by

$$\tilde{d}_{n+1}(a, q, c) = a\delta_{n+1}(1 \otimes q \otimes 1)c - a\rho_n(d_n(\delta_{n+1}(1 \otimes q \otimes 1)))c,$$

for $a, c \in A$, $q \in \mathcal{A}_{n+1}$. The map $\tilde{d}_{n+1}$ is E-multilinear and balanced, and it induces a unique map

$$d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A.$$

The morphism d_{n+1} is in fact a morphism of A-bimodules.

Putting together the equality $\rho_n = s_n + (\rho_n - s_n)$ and Lemmas 3.10 and 3.13, we obtain that $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) = -\rho_n \circ d_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}_n^{\prec}(q)\rangle_R$. Moreover, given $x \in A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$, $x = d_n \circ \rho_n(x) + \rho_{n-1} \circ d_{n-1}(x)$, choosing $x = d_n(\delta_{n+1}(1 \otimes q \otimes 1))$ yields the equality

$$d_n \circ \delta_{n+1}(1 \otimes q \otimes 1) = d_n \circ \rho_n \circ d_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$$

which proves that $d_n \circ d_{n+1} = 0$.

For the proof of (1), fix $q \in \mathcal{A}_n$ and $b \in \mathcal{B}$. Using Lemmas 3.10 and 3.13, we deduce that the element

$$\begin{aligned} &1 \otimes q \otimes \pi(b) - \rho_n \circ d_n(1 \otimes q \otimes \pi(b)) \\ &= 1 \otimes q \otimes \pi(b) - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b)) - \rho_n \circ (d_n - \delta_n)(1 \otimes q \otimes \pi(b)) \end{aligned}$$

differs from $1 \otimes q \otimes \pi(b) - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b))$ by elements in $\langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R$. We will write that

$$(\mathrm{id} - \rho_n \circ \delta_n + \rho_n \circ (d_n - \delta_n))(1 \otimes q \otimes \pi(b)) \equiv \mathrm{id} - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R.$$

Also,

$$\begin{aligned} (\mathrm{id} - \rho_n \circ \delta_n)(1 \otimes q \otimes \pi(b)) &\equiv (\mathrm{id} - s_n \circ \delta_n)(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R \\ &\equiv \delta_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R \\ &\equiv d_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R. \end{aligned}$$

We deduce from this that there exists a unique $\xi \in \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R$ such that

$$(\mathrm{id} - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) + \xi.$$

It is evident that ξ belongs to the kernel of d_n.

The order $\preceq$ satisfies the descending chain condition, so we can use induction on $(k^\times Q_{\geq 0}, \preceq)$. If there is no $\lambda p \in k^\times Q_{\geq 0}$ is such that $\lambda p \prec qb$, then $\xi = 0$ and we define $\rho_{n+1}(1 \otimes q \otimes \pi(b)) = s_{n+1}(1 \otimes q \otimes \pi(b))$. Inductively, suppose that $\rho_{n+1}(\xi)$ is defined. The equality $d_n(\xi) = 0$ implies that $\xi = d_{n+1} \circ \rho_{n+1}(\xi)$ and

$$(\mathrm{id} - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1}(s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi)).$$

We define $\rho_{n+1}(1 \otimes q \otimes \pi(b)) := s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi)$.

Lemmas 3.10 and 3.13 assure that $\rho_{n+1}(\xi)$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n+1}(qb)\rangle_R$, and as a consequence

$$\rho_{n+1}(1 \otimes q \otimes \pi(b)) - s_{n+1}(1 \otimes q \otimes \pi(b)) \in \langle \overline{\mathcal{L}}^{\prec}_{n+1}(qb)\rangle_R.$$

□

We are now ready to prove the theorems.

Proof of Theorem 3.5. We will prove the existence of an $A-E$-bimodule map $\rho_0 : A \otimes_E k\mathcal{A}_{-1} \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ satisfying $d_0 \circ \rho_0 + \rho_{-1} \circ d_{-1} = id$, where $d_{-1} = \mu$ and $\rho_{-1}(a) = s_{-1}(a) = a \otimes 1$ for all $a \in A$. Once this achieved, we apply Proposition 3.14 inductively with $R = k$, for all n such that $0 \leq n \leq N-1$, obtaining this way an homotopy retraction of the complex

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

proving thus that it is exact.

Given $b = b_k \cdots b_1 \in \mathcal{B}$, with $b_i \in Q_1$, $1 \leq i \leq k$,

$$s_0(1 \otimes \pi(b)) = -\sum_i \pi(b_k \cdots b_{k-i+1}) \otimes b_{k-i} \otimes \pi(b_{k-i-1} \cdots b_1).$$

On one hand

$1 \otimes \pi(b) - \pi(b) \otimes 1 = 1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b)))$ and on the other hand the left hand term equals $\delta_0(s_0(1 \otimes \pi(b)))$, yielding $1 \otimes \pi(b) - s_{-1}(1 \otimes \pi(b)) = \delta_0(s_0(1 \otimes \pi(b))$. By hypothesis, $(d_0 - \delta_0)(1 \otimes \pi(b))$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{-1}(b)\rangle_k$, and so there exists $\xi \in \langle \overline{\mathcal{L}}^{\prec}_{-1}(b)\rangle_k$ such that

$$1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b))) = d_0(s_0(1 \otimes \pi(b))) + \xi.$$

It follows that $d_{-1}(\xi) = 0$. Suppose first that there exists no $\lambda p \in k^{\times} Q_{\geq 0}$ such that $\lambda p \prec b$.

In this case $\xi = 0$ and we define $\rho_0(1 \otimes \pi(b)) = s_0(1 \otimes \pi(b))$. Inductively, suppose that $\rho_0(\xi)$ is defined for any ξ such that $d_{-1}(\xi) = 0$. Since in this case $\xi = d_0(\rho_0(\xi))$, we set $\rho_0(1 \otimes \pi(b)) := s_0(1 \otimes \pi(b)) + \rho_0(\xi)$. □

Proof of Theorem 3.6. It follows from the proof of Theorem 3.5 that

$$1 \otimes \pi(b) = (s_{-1} \circ d_{-1} + \delta_0 \circ s_0)(1 \otimes \pi(b))$$

and so $s_{-1} \circ d_{-1} + \delta_0 \circ s_0 = id_{A \otimes_E A}$. Setting $d_0 := \delta_0$, the theorem follows applying Proposition 3.14 for $R = \mathbb{Z}$. □

We end this section by showing that this construction is a generalization of Bardzell's resolution for monomial algebras.

Proposition 3.15. *Given an algebra* A*, let* $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ *be a resolution of* A *as* A*-bimodule such that* $d_\bullet$ *satisfies the hypotheses of Theorem 3.5. If* $p \in \mathcal{A}_n$ *is such that* $r(p) = 0$ *or* $r(p) = p$ *for every reduction* r*, then for all* $a, c \in kQ$*,*

$$d_n(\pi(a) \otimes p \otimes \pi(c)) = \delta_n(\pi(a) \otimes p \otimes \pi(c)).$$

Proof. By hypothesis, we know that there exists no $\lambda' p' \in k^\times Q_{\geq 0}$ such that $\lambda' p' \prec p$, so $\mathcal{L}^{\prec}_{n-1}(p) = \{0\}$ and $d_n(1 \otimes p \otimes 1) = \delta_n(1 \otimes p \otimes 1)$. Given $a, c \in kQ$ we deduce from the previous equality that

$$d_n(\pi(a) \otimes p \otimes \pi(c)) - \delta_n(\pi(a) \otimes p \otimes \pi(c)) = \pi(a)(d_n(1 \otimes p \otimes 1) - \delta_n(1 \otimes p \otimes 1))\pi(c) = 0.$$

□

Corollary 3.16. *Suppose the algebra* $A = kQ/I$ *has a monomial presentation. Choose a reduction system* $\mathcal{R}$ *whose pairs have the monomial relations generating the ideal* I *as first coordinate and* 0 *as second coordinate. In this case, the only maps* d *verifying the hypotheses of Theorem* 4.2 *are those of Bardzell's resolution.*

3.2 Morphisms in low degrees

In this section we describe the morphisms appearing in lower degrees of the resolution.

Let us consider the following data: an algebra $A = kQ/I$ and a reduction system $\mathcal{R}$ satisfying the Diamond condition.

We start by recalling the definition of δ_0 and δ_{-1}. For $a, c \in kQ$, $\alpha \in Q_1$,

$$\begin{aligned} &\delta_{-1} : A \otimes_E A \longrightarrow A, && \delta_0 : A \otimes_E k\mathcal{A}_0 \otimes_E A \longrightarrow A \otimes_E A, \\ &\delta_{-1}(\pi(a) \otimes \pi(c)) = \pi(ac), && \delta_0(\pi(a) \otimes \alpha \otimes \pi(c)) = \pi(a\alpha) \otimes \pi(c) - \pi(a) \otimes \pi(\alpha c). \end{aligned}$$

Definition 3.17. We state some definitions.

- Let $\varphi_0 : kQ \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ be the unique k-linear map such that

$$\varphi_0(c) = \sum_{i=1}^{n} \pi(c_n \cdots c_{i+1}) \otimes c_i \otimes \pi(c_{i-1} \cdots c_1)$$

  for $c \in Q_{\geq 0}$, $c = c_n \cdots c_1$ with $c_i \in Q_1$ for all i, $1 \leq i \leq n$.

- Given a basic reduction $r = r_{a,s,c}$, let $\varphi_1(r, -) : kQ \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ be the unique k-linear map such that, given $p \in Q_{\geq 0}$

$$\varphi_1(r,p) = \begin{cases} \pi(a) \otimes s \otimes \pi(c), & \text{if } p = asc, \\ 0 & \text{if not.} \end{cases} \tag{3.1}$$

  In case $r = (r_n, \dots, r_1)$ is a reduction, where r_i is a basic reduction for all i, $1 \leq i \leq n$, we denote $r' = (r_n, \dots, r_2)$ and we define in a recursive way the map $\varphi_1(r, -)$ as the unique k-linear map from kQ to $A \otimes_E k\mathcal{A}_1 \otimes_E A$ such that

$$\varphi_1(r,p) = \varphi_1(r_1,p) + \varphi_1(r', r_1(p)).$$

- Finally, we define an A-bimodule morphism $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ by the equality

$$d_1(1 \otimes s \otimes 1) = \varphi_0(s) - \varphi_0(\beta(s)), \text{ for all } s \in \mathcal{A}_1.$$

Next we prove four lemmas necessary to the description of the complex in low degrees.

Lemma 3.18. *Let us consider* $p \in Q_{\geq 0}$ *and* $x \in kQ$ *such that* $x \prec p$. *For any reduction* r *the element* $\varphi_1(r, x)$ *belongs to* $\langle \overline{\mathcal{L}_1^{\prec}}(p) \rangle_{\mathbb{Z}}$.

Proof. We will first prove the result for $x = \mu q \in k^{\times} Q_{\geq 0}$. The general case will then follow by linearity. Fix $x = \mu q \in k^{\times} Q_{\geq 0}$. We will use an inductive argument on $(k^{\times} Q_{\geq 0}, \preceq)$.

To start the induction, suppose first that there exists no $\mu' q' \in k^{\times} Q_{\geq 0}$ and that $\mu' q' \prec \mu q = x$. In this case, every basic reduction $r_{a,s,c}$ satisfies either $r_{a,s,c}(x) = x$ or $r_{a,s,c} = 0$. In the first case, $asc \neq q$ and so $\varphi_1(r_{a,s,c}, x) = 0$. In the second case, $asc = q$, so $\varphi_1(r_{a,s,c}, x) = \mu\pi(a) \otimes s \otimes \pi(c)$.

Given an arbitrary reduction $r = (r_n, \dots, r_1)$ with r_i basic for all i, there are three possible cases.

1. $r_1(x) = x$ and $n > 1$,
2. $r_1(x) = x$ and $n = 1$,
3. $r_1(x) = 0$.

Denote $r' = (r_n, \dots, r_2)$ as before and $r_1 = r_{a,s,c}$. In case 1), $\varphi_1(r, x) = \varphi_1(r', x)$. In case 3), $\varphi_1(r, x) = \varphi_1(r_1, x) = 0$. Finally, in case 2), $\varphi_1(r, x) = \varphi_1(r_1, x) = \mu\pi(a) \otimes s \otimes \pi(c)$. Using Lemma 3.8, we obtain that in all three cases $\varphi_1(r, x) \in \langle \overline{\mathcal{L}_1^{\prec}}(p) \rangle_{\mathbb{Z}}$.

Next, suppose that $x = \mu q$ and that the result holds for $\mu' q' \in k^{\times} Q_{\geq 0}$ such that $\mu' q' \prec \mu q = x$. Let us consider r, r_1 and r' as before. Again, there are three possible cases:

1. $asc = q$,
2. $asc \neq q$ and $n > 1$,
3. $asc \neq q$ and $n = 1$.

Case 3) is immediate, since in this situation $\varphi_1(r, x) = 0$. The second case reduces to the other ones, since $\varphi_1(r, x) = \varphi_1(r', x)$. In the first case,

$$\varphi_1(r, x) = \mu\pi(a) \otimes s \otimes \pi(c) + \varphi_1(r', r_1(x)).$$

We know that $r_1(x) \prec x$, and we may write it as a finite sum $r_1(x) = \sum_i \mu_i q_i$. Using the inductive hypothesis, we deduce that $\varphi_1(r, x) \in \langle \overline{\mathcal{L}_1^{\prec}}(p) \rangle_{\mathbb{Z}}$. □

Lemma 3.19. *For all $x \in A \otimes_E k\mathcal{A}_1 \otimes_E A$, x belongs to the kernel of $\delta_0 \circ d_1(x)$.*

Proof. Let x be an element of $A \otimes_E k\mathcal{A}_1 \otimes_E A$. Since these maps are morphisms of A-bimodules, we may suppose $x = 1 \otimes s \otimes 1$, with $s \in \mathcal{A}_1$. A direct computation gives

$$\begin{aligned} \delta_0(d_1(1 \otimes s \otimes 1)) &= \delta_0(\varphi_0(s) - \varphi_0(\beta(s))) \\ &= \pi(s) \otimes 1 - 1 \otimes \pi(s) - \pi(\beta(s)) \otimes 1 + 1 \otimes \pi(\beta(s)) \\ &= 0. \end{aligned}$$

□

Lemma 3.20. *Given $a, c \in Q_{\geq 0}$ and $p = \sum_{i=1}^{n} \lambda_i p_i \in kQ$, with $p_i \in Q_{\geq 0}$ for all i, we obtain the equality*

$$\varphi_0(apc) = \varphi_0(a)\pi(pc) + \pi(a)\varphi_0(p)\pi(c) + \pi(ap)\varphi_0(c).$$

The proof is immediate using the definition of φ_0 and k-linearity of φ_0 and π.

Next we prove the last of the preparatory lemmas.

Lemma 3.21. *Given $p \in Q_{\geq 0}$ and a reduction $r = (r_n, \dots, r_1)$, with r_i a basic reduction for all i such that $1 \leq i \leq n$, there is an equality*

$$d_1(\varphi_1(r_1, p)) = \varphi_0(p) - \varphi_0(r(p)).$$

Proof. We will prove the result by induction on n. We will denote $r_i = r_{a_i, s_i, c_i}$.

For $n = 1$, there are two cases. The first one is when $p \neq a_1 s_1 c_1$. In this situation, $r(p) = r_1(p) = p$, $\varphi_1(r_1, p) = 0$ and so the equality is trivially true. In the second case, $p = a_1 s_1 c_1$, $\varphi_1(r_1, p) = \pi(a_1) \otimes s_1 \otimes \pi(c_1)$ and $r(p) = r_1(p) = a_1 \beta(s_1) c_1$. Moreover,

$$\begin{aligned} d_1(\varphi_1(r_1, p)) + \varphi_0(r_1(p)) &= d_1(\pi(a_1) \otimes s_1 \otimes \pi(c_1)) + \varphi_0(a_1 \beta(s_1) c_1) \\ &= \pi(a_1)\varphi_0(s_1)\pi(c_1) - \pi(a_1)\varphi_0(\beta(s_1))\pi(c_1) + \varphi_0(a_1 \beta(s_1) c_1). \end{aligned}$$

Using Lemma 3.20, the last term equals

$$\varphi_0(a_1)\pi(\beta(s_1)c_1) + \pi(a_1)\varphi_0(\beta(s_1))\pi(c_1) + \pi(a_1\beta(s_1))\varphi_0(c_1),$$

so the whole expression is

$$\begin{aligned} &\pi(a_1)\varphi_0(s_1)\pi(c_1) + \varphi_0(a_1)\pi(\beta(s_1)c_1) + \pi(a_1\beta(s_1))\varphi_0(c_1) \\ &= \pi(a_1)\varphi_0(s_1)\pi(c_1) + \varphi_0(a_1)\pi(s_1 c_1) + \pi(a_1 s_1)\varphi_0(c_1), \end{aligned}$$

and using again Lemma 3.20, this equals $\varphi_0(p)$.

Suppose the result holds for $n - 1$. As usual, we denote $r' = (r_n, \dots, r_2)$.

Since $r(p) = r'(r_1(p))$,

$$\begin{aligned} d_1(\varphi_1(r, p)) + \varphi_0(r(p)) &= d_1(\varphi_1(r_1, p)) + d_1(\varphi_1(r', r_1(p))) + \varphi_0(r'(r_1(p))) \\ &= d_1(\varphi_1(r_1, p)) + \varphi_0(r_1(p)) \\ &= \varphi_0(p). \end{aligned}$$

□

Consider now an element $p \in \mathcal{A}_2$. By definition we write $p = u_0 u_1 u_2 = v_2 v_1 v_0$ where $u_0 u_1$ and $v_1 v_0$ are paths in $\mathcal{A}_1$ dividing p. Suppose $r = r_{a,s,c}$ is a basic reduction such that $r(p) \neq p$. We deduce that either $s = u_0 u_1$ or $s = v_1 v_0$. For an arbitrary reduction $r = (r_n, \dots, r_1)$, we will say that r *starts on the left of* p if $r_1 = r_{a,s,c}$, $s = u_0 u_1$ and $asc = p$, and we will say that r *starts on the right of* p if $r_1 = r_{a,s,c}$, $s = v_1 v_0$ and $asc = p$.

Proposition 3.22. *Let* $\{r^p\}_{p \in \mathcal{A}_2}$ *and* $\{t^p\}_{p \in \mathcal{A}_2}$ *be two sets of reductions such that* $r^p(p)$ *and* $t^p(p)$ *belong to* $k\mathcal{B}$, r^p *starts on the left of* p *and* t^p *starts on the right of* p. *Consider* $d_2 : A \otimes_E k\mathcal{A}_2 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ *the map of* A*-bimodules defined by* $d_2(1 \otimes p \otimes 1) = \varphi_1(t^p, p) - \varphi_1(r^p, p)$.

The sequence

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

is exact.

Proof. To check that d_2 is well defined, consider the map $\tilde{d}_2 : A \times k\mathcal{A}_2 \times A \longrightarrow A \otimes_E k\mathcal{A}_1 \otimes_E A$ defined by $\tilde{d}_2(x,p,y) = x\varphi_1(t^p,p)y - x\varphi_1(r^p,p)y$, for all $x,y \in A$, which is clearly multilinear; taking into account the definition of φ_1, it is such that $\tilde{d}_2(xe,p,y) = \tilde{d}_2(x,ep,y)$ and $\tilde{d}_2(x,pe,y) = \tilde{d}_2(x,p,ey)$ for all $e \in E$, so it induces d_2 on $A \otimes_E k\mathcal{A}_2 \otimes_E A$.

The sequence is a complex:

- $\delta_{-1} \circ \delta_0 = 0$ and $\delta_0 \circ d_1 = 0$ follow from Lemma 3.19.
- Given $p \in \mathcal{A}_2$, $d_1(d_2(1 \otimes p \otimes 1)) = d_1(\varphi_1(t^p,p) - \varphi_1(r^p,p))$. Using Lemma 3.21, this last expression equals $\varphi_0(p) - \varphi_0(t^p(p)) - \varphi_0(p) + \varphi_0(r^p(p))$, which is, by Remark 1.10, equal to $-\varphi_0(\beta(p)) + \varphi_0(\beta(p))$, so $d_1 \circ d_2 = 0$.

It is exact:

- We already know that this is true at A and at $A \otimes_E A$.
- Given $s \in \mathcal{A}_1$, $d_1(1 \otimes s \otimes 1) - \delta_1(1 \otimes s \otimes 1)$ belongs to $\langle \overline{\mathcal{L}_0^{\prec}}(s)\rangle_k$: indeed, notice that $\delta_1(1 \otimes s \otimes 1) = \varphi_0(s)$, and $\varphi_0(\beta(s))$ belongs to $\langle \overline{\mathcal{L}_0^{\prec}}(s)\rangle_k$ since $\beta(s) \prec s$. It follows that

$$d_1(1 \otimes s \otimes 1) - \delta_1(1 \otimes s \otimes 1) = -\varphi_0(\beta(s)) \in \langle \overline{\mathcal{L}_0^{\prec}}(s)\rangle_k.$$

- Given $p \in \mathcal{A}_2$, we will now prove that $(d_2 - \delta_2)(1 \otimes p \otimes 1)$ belongs to $\langle \overline{\mathcal{L}_1^{\prec}}(p)\rangle_k$. We may write $p = u_0u_1u_2 = v_2v_1v_0$, as we did just before this proposition and thus $\delta_2(1 \otimes p \otimes 1) = \pi(v_2) \otimes v_1v_0 \otimes 1 - 1 \otimes u_0u_1 \otimes \pi(u_2)$. Besides, if $r^p = (r_n, \dots, r_1)$ and $t^p = (t_m, \dots, t_1)$ with t_i and r_j basic reductions, the fact that r^p starts on the left and t^p starts on the right of p gives

$$(d_2 - \delta_2)(1 \otimes p \otimes 1) = \varphi_1(t'^p, t_1(p)) - \varphi_1(r'^p, r_1(p)),$$

where $t'^p = (t_m, \dots, t_2)$ and $r'^p = (r_n, \dots, r_2)$. Since $t_1(p) \prec p$ and $r_1(p) \prec p$, Lemma 3.18 allows us to deduce the result.

Finally, Theorem 3.5 implies that the sequence considered is exact. □

Remark 3.23. Given $a \in \mathcal{A}_0 = Q_1$, we have that $\overline{\mathcal{L}_{-1}^{\prec}}(a) = \emptyset$, so for any morphism of A-bimodules $d : A \otimes_E k\mathcal{A}_0 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{-1} \otimes_E A$ such that $(d - \delta_0)(1 \otimes a \otimes 1)$ belongs to $\langle \overline{\mathcal{L}_{-1}^{\prec}}(a)\rangle_k$, it must be $d = \delta_0$.

On the other hand, given $s \in \mathcal{A}_1$, write $\beta(s) = \sum_{i=1}^{m} \lambda_i b_i$. Let $r = r_{a,s',c}$ be a basic reduction such that $r(s) \neq s$. We must have $s' = s$ and $a, c \in Q_0$ must coincide with the source and target of s, respectively. In other words, the only basic reduction such that $r(s) \neq s$ is $r_{a,s,c}$ with a and c as we just said, and in this case $r(s) = \beta(s) \in k\mathcal{B}$.

In this situation

$$\{\lambda q \in k^{\times} Q_{\geq 0} : \lambda q \prec s\} = \{\lambda_1 b_1, \ldots, \lambda_m b_m\},$$

and writing $b_i = b_i^{n_i} \cdots b_i^1$ with $b_i^j \in Q_1$,

$$\overline{\mathcal{L}}_0^{\prec}(s) = \bigcup_{i=1}^{N} \{\lambda_i \pi(b_i^{n_i} \cdots b_i^2) \otimes b_i^1 \otimes 1, \ldots, \lambda_i \otimes b_i^{n_i} \otimes \pi(b_i^{n_i-1} \cdots b_i^1)\}.$$

If $d : A \otimes_E k\mathcal{A}_1 \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_0 \otimes_E A$ verifies $(d - \delta_1)(1 \otimes s \otimes 1) \in \overline{\mathcal{L}}_0^{\prec}(s)$ and $\delta_0 \circ d(s) = 0$ for all $s \in \mathcal{A}_1$, then there exists $\gamma_i^j \in k$ such that

$$d(1 \otimes s \otimes 1) = \varphi_0(s) - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \gamma_i^j \lambda_i \pi(b_i^{n_i} \cdots b_i^{j+1}) \otimes b_i^j \otimes \pi(b_i^{j-1} \cdots b_i^1).$$

From this, applying δ_0 and reordering terms we can deduce that $\gamma_i^j = 1$ for all i, j. We conclude that the unique morphism with the desired properties is d_1.

3.3 Applications

Theorem 3.24. *Given an algebra $A = kQ/I$ such that*

1. *there is a reduction system $\mathcal{R} = \{(s_i, f_i)\}_i$ for I satisfying the Diamond condition with s_i and f_i homogeneous of length $N \geq 2$ for all i,*
2. *for all $n \in \mathbb{N}$, the length of the elements of $\mathcal{A}_n$ is strictly smaller that the length of the elements of $\mathcal{A}_{n+1}$.*

The resolutions of A as A-bimodule obtained using Theorem 3.5 and Theorem 3.6 are minimal.

Proof. Let $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ be a resolution of A as A-bimodule obtained using Theorem 3.5 or Theorem 3.6. Denote by $|c|$ the length of a path $c \in Q_{\geq 0}$. Condition (1) guarantees that for all paths p, q such that $\lambda p \preceq q$ for some $\lambda \in k^{\times}$, we have $|p| = |q|$. Let $n \geq 0$, $q \in \mathcal{A}_n$ and $\lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}_{n-1}^{\prec}(q)$. Since $p \in \mathcal{A}_{n-1}$, condition (2) says that $|p| < |q|$. On the other hand, $\lambda b p b' \prec q$ and then $|bpb'| = |q|$. We deduce that $b \in Q_{\geq 1}$ or $b' \in Q_{\geq 1}$. As a consequence, $\mathrm{Im}(d_n)$ is contained in $J \otimes_E k\mathcal{A}_{n-1} \otimes_E A \cup A \otimes_E k\mathcal{A}_{n-1} \otimes_E J$, where J is the ideal generated by the arrows and therefore the resolution of A is minimal. □

Remark 3.25. The conclusion holds in a more general situation, which includes Example 4.2. It is sufficient to have a reduction system satisfying (1) and such that the ambiguities p that appear when reducing a given $n+1$-ambiguity q are of length strictly smaller than the length of q.

Remark 3.26. In Example 1.27, the reduction system $\mathcal{R}_2$ satisfies the conditions of Theorem 3.24, while $\mathcal{R}_1$ does not satisfy (2).

Notice that if $\mathcal{R}$ is a reduction system for an algebra for which there is a non-resolvable ambiguity, then, even if we complete it like we did in Example 1.27, the resolutions obtained using Theorem 3.5 and Theorem 3.6 will not be minimal.

We end this section proving a generalization of Prop. 8 of [20] and a corollary.

Proposition 3.27. *Let* $A = kQ/I$*, where* Q *is a finite quiver,* kQ *is the path algebra graded by the length of paths and* I *a homogeneous ideal with respect to this grading, contained in* $Q_{\geq 2}$*. Let* $\mathcal{R}$ *be a reduction system satisfying conditions (1) and (2) of Theorem 3.24 and let* A_S *be the associated monomial algebra. The algebra* A_S *is* N*-Koszul if and only if* A *is an* N*-Koszul algebra.*

Proof. The projective bimodules appearing in the minimal resolution of A_S are in one–to–one correspondence with those appearing in the resolution of A, so either both of them are generated in the correct degrees or none is. □

This proposition, together with Thm. 3 of [19] give the following result.

Corollary 3.28. *If* A *has a reduction system* $\mathcal{R}$ *satisfying condition (1) of Theorem 3.24 and such that* $S \subseteq Q_2$*, then* A *is Koszul.*

Chapter 4

Examples

In this chapter we apply the methods developed previously to the following two families of algebras.

- *Quantum complete intersections.* This is the family of algebras $A(\xi, n, m)$ with generators x and y subject to the relations $x^n = 0$, $y^m = 0$ and $yx = \xi xy$, where ξ is an element of the field k and n, m are integers at least equal to 2. These relations are homogeneous if and only if $n = m = 2$, and the algebras $A(\xi, 2, 2)$ are Koszul for all $\xi \in k$. In [10] the authors use these algebras to give a negative answer to Happel's question: if $\xi \in k^\times$ is not a root of unity, then the algebra $A(\xi, 2, 2)$ has finite Hochschild cohomology and infinite global dimension. We begin by studying this subfamily. With our method we recover their Koszul resolution. Then we turn to the general case where our method applies with no further difficulties. This is a nice feature of our method, it treats in a uniform manner algebras of different types. We will see this phenomena again in Chapter 5 with the family of *down-up* algebras.

- *Quantum generalized Weyl algebras.* The members of this family are the algebras $A(a, q)$ with generators y, x, h subject to the relations $hy = qyh, hx = q^{-1}xh, yx = a(h)$ and $xy = a(qh)$, where q is a nonzero element of the field k and $a(h)$ is a polynomial in the variable h. This is an example where we use the procedure explained in Section 1.3 with a non constant weight function ω to find a convenient reduction system.

In both cases we use Proposition 3.22 to find the first degrees of the resolutions and from those formulas we derive the formulas for all degrees.

Before addressing the above examples, we explain how the resolutions obtained using theorems 3.5 and 3.6 depend on the reduction system chosen.

Example 4.1. Consider the algebra $A = k\langle x, y, z\rangle / \left(xyz - x^3 - y^3 - z^3\right)$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$

be the reduction systems

$$\mathcal{R}_1 = \{(z^3, xyz - x^3 - y^3), (xyz^2, x^3z + y^3z + zxyz - zx^3 - zy^3),$$
$$(y^3z^2, -x^3z^2 - z^2xyz + z^2x^3 + z^2y^3 + xyxyz - xyx^3 - xy^4)\}, \text{ and}$$
$$\mathcal{R}_2 = \{(xyz, x^3 + y^3 + z^3\},$$

In examples 1.23 and 1.25 we proved that both $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the Diamond condition. Denote $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ the respective set of n-ambiguities. Notice that $z^{\frac{3}{2}(n+1)} \in \mathcal{A}_n^1$ for n odd and $z^{\frac{3}{2}n+1} \in \mathcal{A}_n^1$ for n even, so $\mathcal{A}_n^1$ is not empty for all $n \in \mathbb{N}$. On the other hand, $\mathcal{A}_n^2$ is empty for all $n \geq 2$. We conclude that using $\mathcal{R}_2$ we will obtain a resolution of length 2, with differentials given explicitely by Proposition 3.22, and using $\mathcal{R}_1$ the resolution obtained will have infinite length. This shows how different the resolutions from different reduction systems can be.

The algebra A is in fact a 3-Koszul algebra. Indeed, denoting by V the k-vector space spanned by x, y, z and by R the one dimensional k-vector space spanned by the relation $xyz - x^3 - y^3 - z^3$, we have

$$R \otimes V \otimes V \cap V \otimes V \otimes R = \{0\},$$

and so the intersection is a subset of $V \otimes R \otimes V$. Theorem 2.5 of [6] guarantees that A is 3-Koszul. By Theorem 3.24, the resolution we obtain from the reduction system $\mathcal{R}_2$ is minimal and therefore it is the Koszul resolution.

4.1 The algebra counterexample to Happel's question

Let ξ be an element of the field k and let A be the k-algebra with generators x and y, subject to the relations $x^2 = 0 = y^2$, $yx = \xi xy$. This algebra can be presented as $k\langle x, y\rangle / \langle X\rangle$, where $X = \{x^2, y^2, yx - \xi xy\}$ and $k\langle x, y\rangle$ is the algebra freely generated in the variables x and y. Choose the order $y < x$ with weights $\omega(x) = \omega(y) = 1$ and define the reduction system $\mathcal{R}_X = \{(x^2, 0), (y^2, 0), (yx, \xi xy)\}$ as explained in Remark 1.18. By Proposition 1.24 the reduction system $\mathcal{R}_X$ satisfies the Diamond condition. The details are given in Example 1.25.

Hence, the set of irreducible paths is $\mathcal{B} = \{1, x, y, xy\}$. The only path of length 2 not in $S_{\mathcal{R}_X}$ is xy and Proposition 1.40 implies that for each n, $\mathcal{A}_n$ is the set of paths of lenght $n+1$ not divisible by xy,

$$\mathcal{A}_n = \{y^s x^t : s + t = n + 1\}.$$

Lemma 4.2. *The following complex provides the beginning of an A-bimodule projective resolution of the algebra A*

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where d_1 *is the* A*-bimodule map such that*

$$d_1(1 \otimes x^2 \otimes 1) = x \otimes x \otimes 1 + 1 \otimes x \otimes x,$$
$$d_1(1 \otimes y^2 \otimes 1) = y \otimes y \otimes 1 + 1 \otimes y \otimes y,$$
$$d_1(1 \otimes yx \otimes 1) = y \otimes x \otimes 1 + 1 \otimes y \otimes x - \xi x \otimes y \otimes 1 - \xi \otimes x \otimes y$$

and d_2 *is the* A*-bimodule morphism such that*

$$d_2(1 \otimes y^3 \otimes 1) = y \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes y,$$
$$d_2(1 \otimes y^2x \otimes 1) = y \otimes yx \otimes 1 + \xi \otimes yx \otimes y + \xi^2 x \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes x,$$
$$d_2(1 \otimes yx^2 \otimes 1) = y \otimes x^2 \otimes 1 - 1 \otimes yx \otimes x - \xi x \otimes yx \otimes 1 - \xi^2 \otimes x^2 \otimes y$$
$$d_2(1 \otimes x^3 \otimes 1) = x \otimes x^2 \otimes 1 - 1 \otimes x^2 \otimes x.$$

Proof. We apply Proposition 3.22 to the following sets $\{r^p\}_{p\in\mathcal{A}_2}$ of left reductions and $\{t^p\}_{p\in\mathcal{A}_2}$ of right reductions, where

$$\begin{aligned} r^{y^3} &= r_{1,y^2,y}, & r^{y^2x} &= r_{1,y^2,x}, \\ r^{yx^2} &= (r_{1,x^2,y}, r_{x,yx,1}, r_{1,yx,x}), & r^{x^3} &= r_{1,x^2,x}, \\ t^{y^3} &= r_{y,y^2,1}, & t^{y^2x} &= (r_{x,y^2,1}, r_{1,yx,y}, r_{y,yx,1}), \\ t^{yx^2} &= r_{y,x^2,1}, & t^{x^3} &= r_{x,x^2,1}. \end{aligned}$$

□

One can find an A-bimodule resolution of A in [10] and in [8]; the authors also compute the Hochschild cohomology of A therein. We recover this resolution with our method.

Given $q \in \mathcal{A}_n$, there are $s, t \in \mathbb{N}$ such that $s + t = n + 1$ and $q = y^s x^t$. Suppose $q = apc$ with $p = y^{s'}x^{t'} \in \mathcal{A}_{n-1}$ and $a, c \in Q_{\geq 0}$. Since $s + t = n + 1$ and $s' + t' = n$, either a belongs to Q_0 and $c = x$ or $a = y$ and $c \in Q_0$. As a consequence of this fact, the maps

$\delta_n : kQ \otimes_E k\mathcal{A}_n \otimes_E kQ \longrightarrow kQ \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ are

$$\delta_n(1 \otimes y^s x^t \otimes 1) = \begin{cases} y \otimes y^{s-1}x^t \otimes 1 + (-1)^{n+1} \otimes y^s x^{t-1} \otimes x, & \text{if } s \neq 0 \text{ and } t \neq 0, \\ y \otimes y^n \otimes 1 + (-1)^{n+1} \otimes y^n \otimes y, & \text{if } t = 0, \\ x \otimes x^n \otimes 1 + (-1)^{n+1} \otimes x^n \otimes x, & \text{if } s = 0, \end{cases} \tag{4.1}$$

Moreover, given a basic reduction $r = r_{a,s,c}$, the fact that s belongs to $S = \{x^2, y^2, yx\}$ implies that $r(y^s x^t)$ is either 0 or $\xi y^{s-1}xyx^{t-1}$. Considering the reduction system $\mathcal{R}$, if $s \neq 0$ and $t \neq 0$, then

$$\overline{\mathcal{L}}^{\prec}_{n-1}(y^s x^t) = \{\xi^s x \otimes y^s x^{t-1} \otimes 1, \xi^t \otimes y^{s-1}x^t \otimes y\}.$$

In case $s = 0$ or $t = 0$, the set $\overline{\mathcal{L}}^{\prec}_{n-1}(y^s x^t)$ is empty.

The computation of $d_2 - \delta_2$ suggests the definition of the maps

$$d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$$

as follows

$$d_n(1 \otimes y^s x^t \otimes 1) = \delta_n(1 \otimes y^s x^t \otimes 1) + \epsilon(\xi^s x \otimes y^s x^{t-1} \otimes 1 + \xi^t \otimes y^{s-1} x^t \otimes y)$$

where ϵ denotes a sign depending on s, t, n. The equality $d_{n-1} \circ d_n = 0$ shows that making the choice $\epsilon = (-1)^s$ does the job.

Finally, Theorem 3.5 shows that the complex

$$\cdots \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A \xrightarrow{d_n} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

with

$$\begin{aligned} d_n(1 \otimes y^s x^t \otimes 1) =& y \otimes y^{s-1} x^t \otimes 1 + (-1)^{n+1} 1 \otimes y^s x^{t-1} \otimes x \\ &+ (-1)^s \xi^s x \otimes y^s x^{t-1} \otimes 1 + (-1)^s \xi^t \otimes y^{s-1} x^t \otimes y, \end{aligned}$$

for $s > 0$ and $t > 0$, and

$$\begin{aligned} d_n(1 \otimes y^{n+1} \otimes 1) &= y \otimes y^n \otimes 1 + (-1)^{n+1} 1 \otimes y^n \otimes y, \\ d_n(1 \otimes x^{n+1} \otimes 1) &= x \otimes x^n \otimes 1 + (-1)^{n+1} 1 \otimes x^n \otimes x, \end{aligned}$$

is a projective bimodule resolution of A.

Again, the algebra A is Koszul, see for example [7] and the resolution obtained using our procedure is the Koszul resolution, which is the minimal one, see Theorem 3.24.

4.2 Quantum complete intersections

These algebras generalize the previous case. Instead of the relations $x^2 = 0 = y^2$, $yx = \xi xy$, we have $x^n = 0 = y^m$, $yx = \xi xy$, where n and m are fixed positive integers, $n, m > 1$. We still denote the algebra by A and similarly to the previous case, let $X = \{x^n, y^m, yx - \xi xy\}$. Consider the order $y < x$ with weights $\omega(x) = \omega(y) = 1$. The reduction system $\mathcal{R}_X = \{(x^n, 0), (y^m, 0), (yx, \xi xy)\}$ satisfies conditions $1, 2'$ and 3 of remarks 1.18 and 1.19. Similar calculations to the previous case show that every ambiguity of $\mathcal{R}_X$ is resolvable. By Proposition 1.24, this implies that $\mathcal{R}_X$ satisfies the Diamond condition.

The set of irreducible paths is $\mathcal{B} = \{x^i y^j \in k\langle x, y\rangle : 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$. By direct computations we obtain $\mathcal{A}_1 = \{y^m, yx, x^n\}$, $\mathcal{A}_2 = \{y^{m+1}, y^m x, yx^n, x^{n+1}\}$ and

$\mathcal{A}_3 = \{y^{2m}, y^{m+1}x, y^m x^n, yx^{n+1}, x^{2n}\}$. To obtain a general formula for $\mathcal{A}_N$, denote by $\varphi : \mathbb{N}_0^2 \longrightarrow \mathbb{N}_0$ the map

$$\varphi(s,n) = \begin{cases} \frac{s}{2}n & \text{if } s \text{ is even,} \\ \\ \frac{s-1}{2}n+1 & \text{if } s \text{ is odd.} \end{cases} \tag{4.2}$$

Hence, the set of N-ambiguities is $\mathcal{A}_N = \{y^{\varphi(s,m)}x^{\varphi(t,n)} : s+t = N+1\}$. We will sometimes write (s,t) instead of $y^{\varphi(s,m)}x^{\varphi(t,n)}$.

We first compute the beginning of the resolution.

Lemma 4.3. *The following complex provides the beginning of a projective resolution of* A *as* A*-bimodule:*

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where d_1 *and* d_2 *are morphisms of* A*-bimodules given by the formulas*

$$d_1(1 \otimes x^n \otimes 1) = \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-1-i},$$
$$d_1(1 \otimes y^m \otimes 1) = \sum_{i=0}^{m-1} y^i \otimes y \otimes y^{m-1-i},$$
$$d_1(1 \otimes yx \otimes 1) = 1 \otimes y \otimes x + y \otimes x \otimes 1 - \xi \otimes x \otimes y - \xi x \otimes y \otimes 1$$
$$d_2(1 \otimes y^{m+1} \otimes 1) = y \otimes y^m \otimes 1 - 1 \otimes y^m \otimes y,$$
$$d_2(1 \otimes y^m x \otimes 1) = \sum_{i=0}^{m-1} \xi^i y^{m-1-i} \otimes yx \otimes y^i + \xi^m x \otimes y^m \otimes 1 - 1 \otimes y^m \otimes x$$
$$d_2(1 \otimes yx^n \otimes 1) = y \otimes x^n \otimes 1 - \sum_{i=0}^{n-1} \xi^i x^i \otimes yx \otimes x^{n-1-i} - \xi^n \otimes x^n \otimes y,$$
$$d_2(1 \otimes x^{n+1} \otimes 1) = x \otimes x^n \otimes 1 - 1 \otimes x^n \otimes x.$$

Proof. It is straightforward, using Proposition 3.22 applied to the set $\{r^p\}_{p \in \mathcal{A}_2}$ of left reductions, where

$$r^{y^{m+1}} = r_{1,y^m,y}, \qquad r^{y^m x} = r_{1,y^m,x},$$
$$r^{yx^n} = (r_{1,x^n,y}, \dots, r_{x,yx,x^{n-2}}, r_{1,yx,x^{n-1}}) \qquad r^{x^{n+1}} = r_{1,x^n,x},$$

and the set $\{t^p\}_{p \in \mathcal{A}_2}$ of right reductions, where

$$t^{y^{m+1}} = r_{y,y^m,1}, \quad t^{y^m x} = (r_{x,y^m,1}, \dots, r_{y^{m-2},yx,y}, r_{y^{m-1},yx,1}),$$
$$y^{yx^n} = r_{y,x^n,1}, \quad t^{x^{n+1}} = r_{x,x^n,1}.$$

□

Next we proceed to construct the rest of the resolution. Denote $(s,t) = y^{\varphi(s,m)}x^{\varphi(t,n)} \in \mathcal{A}_N$. We will first describe the set $\overline{\mathcal{L}}^{\prec}_{N-1}(s,t)$. There are four cases, depending on the parity of s, t and N. With this in view, it is useful to make some previous computations that we list below.

1. For s even, for all j, $0 \leq j \leq m-1$, $y^{\varphi(s,m)} = y^{m-1-j}y^{\varphi(s-1,m)}y^j$.
2. For s odd, $y^{\varphi(s,m)} = yy^{\varphi(s-1,m)} = y^{\varphi(s-1,m)}y$.
3. For t even, for all i, $0 \leq i \leq n-1$, $x^{\varphi(t,n)} = x^i x^{\varphi(t-1,n)}x^{n-i-1}$,
4. For t odd, $x^{\varphi(t,n)} = xx^{\varphi(t-1,n)} = x^{\varphi(t-1,n)}x$.

First case N even, s even, t odd,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)j}y^{m-1-j} \otimes (s-1,t) \otimes y^j\}_{j=1}^{m-1} \cup \{\xi^{\varphi(s,m)}x \otimes (s,t-1) \otimes 1\}.$$

Second case N even, s odd, t even,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y\} \cup \{\xi^{\varphi(s,m)i}x^i \otimes (x,t-1) \otimes x^{n-1-i}\}_{i=1}^{n-1}.$$

Third case N odd, s even, t even,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)j}y^{m-1-j} \otimes (s-1,t) \otimes y^j\}_{j=1}^{m-1} \cup \{\xi^{\varphi(s,m)i}x^i \otimes (s,t-1) \otimes x^{n-1-i}\}_{i=1}^{n-1}.$$

Fourth case N, s and t odd,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)}1 \otimes (s-1,t) \otimes y, \xi^{\varphi(s,m)}x \otimes (s,t-1) \otimes 1\}.$$

Remark 4.4. We observe that, analogously to the case $n = m = 2$,

$$(d_1 - \delta_1)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_0(s,t)} u,$$
$$(d_2 - \delta_2)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_1(s,t)} u.$$

Proposition 3.14 for $R = \mathbb{Z}$ guarantees that there exist A-bimodule maps $d_N : A \otimes_E k\mathcal{A}_N \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{N-1} \otimes_E A$ such that $(d_N - \delta_N)(1 \otimes (s,t) \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{N-1}(s,t) \rangle_{\mathbb{Z}}$ and, most important, the complex $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ is a projective resolution of A as A-bimodule.

We are not yet able at this point to give the explicit formulas of the differentials.

In order to illustrate the situation, let us describe what happens for $N = 3$. We know after the mentioned proposition that there exist $t_1, t_2 \in \mathbb{Z}$ such that

$$\begin{aligned} d_3(1 \otimes y^{m+1}x \otimes 1) &= d_3(1 \otimes (3,1) \otimes 1) \\ &= \delta_3(1 \otimes (3,1) \otimes 1) + t_1\xi \otimes (2,1) \otimes y + t_2\xi^3 x \otimes (3,0) \otimes 1 \\ &= y \otimes y^m x \otimes 1 + 1 \otimes y^{m+1} \otimes x + t_1\xi \otimes y^m x \otimes y + t_2\xi^3 x \otimes y^{m+1} \otimes 1. \end{aligned}$$

Of course, $d_2 \circ d_3 = 0$. It follows from this equality that $t_1 = t_2 = -1$. This example motivates the following lemma, stated in terms of the preceding notations.

Lemma 4.5. *The* A*-bimodule morphisms* $d_N : A \otimes_E k\mathcal{A}_N \otimes_E A \longrightarrow A \otimes_E k\mathcal{A}_{N-1} \otimes_E A$ *defined by the formula*

$$d_N(1 \otimes (s,t) \otimes 1) = \delta_N(1 \otimes (s,t) \otimes 1) + (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_{N-1}(s,t)} u$$

satisfy the hypotheses of Thm. 3.5.

Proof. It is straightforward. □

We gather all the information we have obtained about the projective bimodule resolution of A in the following proposition.

Proposition 4.6. *The complex of* A*-bimodules* $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$*, with*

$$\mathcal{A}_N = \{y^{\varphi(s,m)} x^{\varphi(t,n)} : s + t = N + 1\}$$

and differentials defined as follows is exact.

1. *For* N *even,* s *even and* t *odd,*

$$\begin{aligned} d_N(1 \otimes (s,t) \otimes 1) &= y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^s \xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j \\ &+ (-1)^{N+1} 1 \otimes (s,t-1) \otimes x + (-1)^s \xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1. \end{aligned}$$

2. *For* N *even,* s *odd and* t *even,*

$$\begin{aligned} d_N(1 \otimes (s,t) \otimes 1) &= y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y \\ &+ (-1)^{N+1} 1 \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{\varphi(s,m)i} x^i \otimes (s,t-1) \otimes x^{n-1-i} \end{aligned}$$

3. *For* N *odd,* s *and* t *even,*

$$\begin{aligned} d_N(1 \otimes (s,t) \otimes 1) &= y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^s \xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j \\ &+ (-1)^{N+1} 1 \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{\varphi(s,m)i} x^i \otimes (s,t-1) \otimes x^{n-1-i} \end{aligned}$$

4. *For* N*,* s *and* t *odd,*

$$\begin{aligned} d_N(1 \otimes (s,t) \otimes 1) &= y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y \\ &+ (-1)^{N+1} 1 \otimes (s,t-1) \otimes x + (-1)^s \xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1. \end{aligned}$$

Again, we obtain the minimal resolution of A, even for $n \neq 2$ or $m \neq 2$, when the algebra is not homogeneous.

4.3 Quantum generalized Weyl algebras

Let k be a field, $q \in k^{\times}$ and $a \in k[h]$. Denote A the k-algebra with generators y, x, h, subject to the relations

$$hy = qyh, \qquad hx = q^{-1}xh, \qquad yx = a(h), \qquad xy = a(qh),$$

where $a(h) \in k[h]$. Write $a(h) = \sum_{i=0}^{N} a_i h^i$ with $a_N \neq 0$. Let X be the set $\{hy - qyh, hx - q^{-1}xh, xy - a(h), yx - a(qh)\}$ contained in the free algebra $k\langle x, y, h\rangle$. By definition $A = k\langle x, y, h\rangle / \langle X\rangle$. Consider the order $h < y < x$ and weights $\omega(h) = 1, \omega(x) = 1$ and $\omega(y) = N$. Observe that

$$hy > yh, \qquad hx > xh, \qquad xy > a(h), \qquad yx > a(qh).$$

All the ambiguities of the reduction system $\mathcal{R}_X = \{(hy, qyh), (hx, q^{-1}xh), (xy, a(h)), (yx, a(qh))\}$ are resolvable. Proposition 1.24 implies that $\mathcal{R}_X$ satisfies the Diamond condition. Notice that $\mathcal{S} = \{hy, hx, xy, yx\}$ and therefore the set of paths of length two not in S is $\{yy, yh, xx, xh, hh\}$. By Proposition 1.40, the sets of n-ambiguities, for $n \geq 1$, are

- for n even, $\mathcal{A}_n = \{h(yx)^{\frac{n}{2}}, h(xy)^{\frac{n}{2}}, x(yx)^{\frac{n}{2}}, y(xy)^{\frac{n}{2}}\}$,
- for n odd, $\mathcal{A}_n = \{h(yx)^{\frac{n-1}{2}}y, h(xy)^{\frac{n-1}{2}}x, (xy)^{\frac{n+1}{2}}, (yx)^{\frac{n+1}{2}}\}$.

Lemma 4.7. *The following complex provides the beginning of an A-bimodule projective resolution of the algebra A*

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where d_1 is the A-bimodule map such that

$$d_1(1 \otimes hy \otimes 1) = 1 \otimes h \otimes y + h \otimes y \otimes 1 - q \otimes y \otimes h - qy \otimes h \otimes 1,$$
$$d_1(1 \otimes hx \otimes 1) = 1 \otimes h \otimes x + h \otimes x \otimes 1 - q^{-1} \otimes x \otimes h - q^{-1}x \otimes h \otimes 1,$$
$$d_1(1 \otimes xy \otimes 1) = 1 \otimes x \otimes y + x \otimes y \otimes 1 - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^i h^{i-k-1} \otimes h \otimes h^k,$$
$$d_1(1 \otimes yx \otimes 1) = 1 \otimes y \otimes x + y \otimes x \otimes 1 - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i h^{i-k-1} \otimes h \otimes h^k,$$

and d_2 is the A-bimodule morphism such that

$$d_2(1 \otimes hyx \otimes 1) = h \otimes yx \otimes 1 - 1 \otimes hy \otimes x - qy \otimes hx \otimes 1 - 1 \otimes yx \otimes h,$$
$$d_2(1 \otimes hxy \otimes 1) = h \otimes xy \otimes 1 - 1 \otimes hx \otimes y - q^{-1}x \otimes hy \otimes 1 - 1 \otimes xy \otimes h,$$
$$d_2(1 \otimes xyx \otimes 1) = x \otimes yx \otimes 1 - 1 \otimes xy \otimes x - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^{i-k} h^{i-k-1} \otimes hx \otimes h^k,$$
$$d_2(1 \otimes yxy \otimes 1) = y \otimes xy \otimes 1 - 1 \otimes yx \otimes y - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^k h^{i-k-1} \otimes hy \otimes h^k.$$

Proof. It is a consequence of Proposition 3.22 applied to the only possible set of reductions starting on the left and on the right of p for every $p \in \mathcal{A}_2$. $\square$

Now we proceed to find a complete resolution of A. Let $n \geq 3$ and $q \in \mathcal{A}_n$. We want to describe the morphisms δ_n and the sets $\overline{\mathcal{L}}^{\prec}_{n-1}(q)$. Observe that every $n-1$-ambiguity has either no letter h or it has only one letter h at the left, and every $n-1$-ambiguity has length n. Also, notice that every reduction moves a letter h from left to right or replaces, respectively, yx or xy by $a(h)$ or $a(qh)$. As a consequence, the morphism $\delta_n : A \otimes_k k\mathcal{A}_n \otimes_k A \longrightarrow A \otimes_k k\mathcal{A}_{n-1} \otimes_k A$ is

- If n is even, set $l := \frac{n}{2}$. Then

$$\begin{aligned}
\delta_n(1 \otimes h(yx)^l \otimes 1) &= h \otimes (yx)^l \otimes 1 - 1 \otimes h(yx)^{l-1}y \otimes x,\\
\delta_n(1 \otimes h(xy)^l \otimes 1) &= h \otimes (xy)^l \otimes 1 - 1 \otimes h(xy)^{l-1}x \otimes y,\\
\delta_n(1 \otimes x(yx)^l \otimes 1) &= x \otimes (yx)^l \otimes 1 - 1 \otimes (xy)^l \otimes x,\\
\delta_n(1 \otimes y(xy)^l \otimes 1) &= y \otimes (xy)^l \otimes 1 - 1 \otimes (yx)^l \otimes y.
\end{aligned}$$

- If n odd, set $l := \frac{n-1}{2}$. Then

$$\begin{aligned}
\delta_n(1 \otimes h(yx)^l y \otimes 1) &= h \otimes (yx)^l y \otimes 1 + 1 \otimes h(yx)^l \otimes y,\\
\delta_n(1 \otimes h(xy)^l x \otimes 1) &= h \otimes (xy)^l x \otimes 1 + 1 \otimes h(xy)^l \otimes x,\\
\delta_n(1 \otimes (xy)^{l+1} \otimes 1) &= x \otimes y(xy)^l \otimes 1 + 1 \otimes x(yx)^l \otimes y,\\
\delta_n(1 \otimes (yx)^{l+1} \otimes 1) &= y \otimes x(yx)^l \otimes 1 + 1 \otimes y(xy)^l \otimes x.
\end{aligned}$$

Moreover, the sets $\overline{\mathcal{L}}^{\prec}_{n-1}(q)$ are as follows.

- If n is even, set $l := \frac{n}{2}$. Then

$$\begin{aligned}
\overline{\mathcal{L}}^{\prec}_{n-1}(h(yx)^l) &= \{qy \otimes h(xy)^{l-1}x \otimes 1, 1 \otimes (yx)^l \otimes h\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}(h(xy)^l) &= \{q^{-1}x \otimes h(yx)^{l-1}y \otimes 1, 1 \otimes (xy)^l \otimes h\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}(x(yx)^l) &= \{a_i q^{i-k} h^{i-k-1} \otimes h(xy)^{l-1}x \otimes h^k : 1 \leq i \leq N, 0 \leq k \leq i-1\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}(y(xy)^l) &= \{a_i q^k h^{i-k-1} \otimes h(yx)^{l-1}y \otimes h^k : 1 \leq i \leq N, 0 \leq k \leq i-1\}.
\end{aligned}$$

- If n is odd, set $l = \frac{n-1}{2}$. Then

$$\begin{aligned}
\overline{\mathcal{L}}^{\prec}_{n-1}(h(yx)^l y) &= \{qy \otimes h(xy)^l \otimes 1, q \otimes y(xy)^l \otimes h\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}(h(xy)^l x) &= \{q^{-1}x \otimes h(yx)^l \otimes 1, q^{-1} \otimes x(yx)^l \otimes h\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}((xy)^{l+1}) &= \{a_i q^i h^{i-k-1} \otimes h(xy)^l \otimes h^k : 1 \leq i \leq N, 0 \leq k \leq i-1\},\\
\overline{\mathcal{L}}^{\prec}_{n-1}((yx)^{l+1}) &= \{a_i h^{i-k-1} \otimes h(yx)^l \otimes h^k : 1 \leq i \leq N, 0 \leq k \leq i-1\}.
\end{aligned}$$

By Theorem 3.6 there exist A-bimodule morphisms $d_n : A \otimes_k k\mathcal{A}_n \otimes_k A \longrightarrow A \otimes_k k\mathcal{A}_{n-1} \otimes_k A$ such that $d_{n-1} \circ d_n = 0$ for $n \geq 0$ and $d_n(q) - \delta_n(q) \in \langle \overline{\mathcal{L}}^{\prec}_{n-1}(q) \rangle$ for all $n \geq -1$ and $q \in \mathcal{A}_n$. We have formulas for the differentials in low degrees and, as we have already done in the previous examples, we can use these formulas to get an idea of how can the following differentials be. Fix for example some n odd, $n \geq 3$ and consider $q = h(yx)^l y$, where $l = \frac{n-1}{2}$. By Theorem 3.6 there exist integers a, b such that

$$d_n(1 \otimes h(yx)^l y \otimes 1) = \delta_n(1 \otimes h(yx)^l y \otimes 1) + aqy \otimes h(xy)^l \otimes 1 + bq \otimes (yx)^l y \otimes h.$$

Using the formula we have for δ_n and reordering the terms, this equality becomes

$$\begin{aligned} &d_n(1 \otimes h(yx)^l y \otimes 1 = \\ &= 1 \otimes h(yx)^l \otimes y + h \otimes (yx)^l y \otimes 1 + bq \otimes (yx)^l y \otimes h + aqy \otimes h(xy)^l \otimes 1. \end{aligned} \tag{4.3}$$

Looking at the left and right factors in each term of the right side, we find that this equality is very similar to a formula we already have:

$$d_1(1 \otimes hy \otimes 1) = 1 \otimes h \otimes y + h \otimes y \otimes 1 - q \otimes y \otimes h - qy \otimes h \otimes 1,$$

which corresponds to $n = 1$ and $l = 0$ in (4.3). This suggests to set $a = b = -1$. For every $q \in \mathcal{A}_n$ there is a similar argument which leads to the following Lemma.

Lemma 4.8. *Consider the* A*-bimodule morphisms* $d_n : A \otimes_k k\mathcal{A}_n \otimes_k A \longrightarrow A \otimes_k k\mathcal{A}_{n-1} \otimes_k A$ *of Lemma 4.7 for* $n \leq 2$ *and the following formulas for* $n \geq 3$,

- *If* n *is even and* $l = \frac{n}{2}$,

$$d_n(1 \otimes h(yx)^l \otimes 1) = h \otimes (yx)^l \otimes 1 - 1 \otimes h(yx)^{l-1} y \otimes x - qy \otimes h(xy)^{l-1} x \otimes 1 - 1 \otimes (yx)^l \otimes h,$$
$$d_n(1 \otimes h(xy)^l \otimes 1) = h \otimes (xy)^l \otimes 1 - 1 \otimes h(xy)^{l-1} x \otimes y - q^{-1} x \otimes h(yx)^{l-1} y \otimes 1 - 1 \otimes (xy)^l \otimes h,$$
$$d_n(1 \otimes x(yx)^l \otimes 1) = x \otimes (yx)^l \otimes 1 - 1 \otimes (xy)^l \otimes x - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^{i-k} h^{i-k-1} \otimes h(xy)^{l-1} x \otimes h^k,$$
$$d_n(1 \otimes y(xy)^l \otimes 1) = y \otimes (xy)^l \otimes 1 - 1 \otimes (yx)^l \otimes y - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^k h^{i-k-1} \otimes h(yx)^{l-1} y \otimes h^k.$$

- *If* n *is odd and* $l = \frac{n-1}{2}$,

$$d_n(1 \otimes h(yx)^l y \otimes 1) = h \otimes y(xy)^l \otimes 1 + 1 \otimes h(yx)^l \otimes y - qy \otimes h(xy)^l \otimes 1 - q \otimes y(xy)^l \otimes h,$$
$$d_n(1 \otimes h(xy)^l x \otimes 1) = h \otimes x(yx)^l \otimes 1 + 1 \otimes h(xy)^l \otimes x - q^{-1} x \otimes h(yx)^l \otimes 1 - q^{-1} \otimes x(yx)^l \otimes h,$$
$$d_n(1 \otimes (xy)^{l+1} \otimes 1) = x \otimes y(xy)^l \otimes 1 + 1 \otimes x(yx)^l \otimes y - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i q^i h^{i-k-1} \otimes h(xy)^l \otimes h^k,$$
$$d_n(1 \otimes (yx)^{l+1} \otimes 1) = y \otimes x(yx)^l \otimes 1 + 1 \otimes y(xy)^l \otimes x - \sum_{i=1}^{N} \sum_{k=0}^{i-1} a_i h^{i-k-1} \otimes h(yx)^l \otimes h^k.$$

For all $n \geq 0$*, these morphisms verify* $d_{n-1} \circ d_n = 0$ *and* $(d_n - \delta_n)(1 \otimes q \otimes 1) \in \langle \overrightarrow{\mathcal{L}}_{n-1}^{\prec}(q) \rangle_{\mathbb{Z}}$ *for all* $q \in \mathcal{A}_n$

Proof. Lemma 4.7 gives the result for $0 \leq n \leq 2$. Check by hand that $d_{n-1} \circ d_n = 0$ for all $n \geq 3$. The second part of the Lemma follows from the previous discussion. □

Theorem 4.9. *The complex* $(A \otimes_k k\mathcal{A}_\bullet \otimes_k A, d_\bullet)$*, where* $d_\bullet$ *are the differentials of the previous lemma, is a projective resolution of* A *by* A*-bimodules.*

Proof. Use Lemma 4.8 and Theorem 3.5. □

Chapter 5

Down-up algebras

In this chapter we state and prove some results we obtained about down-up algebras. We recall their definition. Let k be a field and $\alpha, \beta, \gamma \in k$. The down-up algebra $A(\alpha, \beta, \gamma)$ is the quotient of $k\langle d, u\rangle$ by the two sided ideal I generated by relations

$$d^2u - \alpha dud - \beta ud^2 - \gamma d = 0,$$
$$du^2 - \alpha udu - \beta u^2d - \gamma u = 0.$$

Down-up algebras have been deeply studied since they were defined in [12]. We can mention the articles [16], [14],[9], [17], [15], [23], [24], [26], [27], [28], [29], [30], in which the authors prove diverse properties of down-up algebras. We recall some of them.

- The algebra $A(\alpha, \beta, \gamma)$ is noetherian if and only if $\beta \neq 0$ [24]

- Down-up algebras are graded with $\mathrm{dg}(d) = 1$, $\mathrm{dg}(u) = -1$, and they are filtered if we consider d and u of weight 1. If $\gamma = 0$ they are also graded by this weight.

- $A(\alpha, \beta, \gamma)$ is 3-Koszul if and only if $\gamma = 0$, and if $\gamma \neq 0$, it is a PBW deformation of a 3-Koszul algebra [9].

We organize our results in four sections. In Section 5.1 we use the methods developed in this thesis to obtain a lenght three resolution of the algebra $A(\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in k$, which we will use in the other three sections. In Section 5.2 we find which down-up algebras are twisted 3-Calabi-Yau and which of them are 3-Calabi-Yau. In Section 5.3 we show that the algebra $A(\alpha, \beta, \gamma)$ is monomial if and only if $\alpha = \beta = \gamma = 0$. This result shows that the resolution obtained in Section 5.1 can't be obtained using Bardzell's methods. In Section 5.4 we solve the isomorphism problem for the non noetherian down-up algebras. The isomorphism problem for noetherian down-up algebras was solved in [16].

5.1 The resolution

Let k be a field and $\alpha, \beta, \gamma \in k$. Denote $A = A(\alpha, \beta, \gamma)$.

Let Q be the quiver with one vertex and two arrows d, u. Fix a lexicographical order such that $d < u$, with weights $\omega(d) = 1 = \omega(u)$. The reduction system $\mathcal{R} = \{(d^2u, \alpha dud + \beta ud^2 + \gamma d), (du^2, \alpha udu + \beta u^2 d + \gamma u)\}$ has $\mathcal{B} = \{u^i(du)^k d^j : i, k, j \in \mathbb{N}_0\}$ as set of irreducible paths and $\mathcal{A}_2 = \{d^2u^2\}$; using Bergman's Diamond Lemma we see that $\mathcal{R}$ satisfies condition $(\lozenge)$. Also, $\mathcal{A}_0 = \{d, u\}$ and $\mathcal{A}_n = \emptyset$ for all $n \geq 3$. The set $\mathcal{B}$ is the k-basis already considered in [12].

Proposition 5.1. *The following sequence is a free resolution of* A *as* A*-bimodule:*

$$0 \longrightarrow A \otimes kd^2u^2 \otimes A \xrightarrow{d_2} A \otimes (kd^2u \oplus kdu^2) \otimes A \xrightarrow{d_1} A \otimes (kd \oplus ku) \otimes A \xrightarrow{\delta_0} A \otimes A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where $\otimes$ *denotes* $\otimes_k$ *and*

$$\begin{aligned} d_1(1 \otimes d^2u \otimes 1) &= 1 \otimes d \otimes du + d \otimes d \otimes u + d^2 \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes d^2 + u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1, \\ d_1(1 \otimes du^2 \otimes 1) &= 1 \otimes d \otimes u^2 + d \otimes u \otimes u + du \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes ud + u \otimes u \otimes d + u^2 \otimes d \otimes 1) - \gamma \otimes u \otimes 1, \end{aligned}$$

and

$$d_2(1 \otimes d^2u^2 \otimes 1) = d \otimes du^2 \otimes 1 + \beta \otimes du^2 \otimes d - 1 \otimes d^2u \otimes u - \beta u \otimes d^2u \otimes 1.$$

We denote this resolution by C.

Proof. The reductions $r^{d^2u^2} = (r_{u,d^2u,1}, r_{1,d^2u,u})$ and $t^{d^2u^2} = (t_{1,du^2,d}, t_{d,du^2,1})$ are respectively left and right reductions of d^2u^2. Recall that $\mathcal{A}_n = \emptyset$ for all $n \geq 3$. The result follows from Proposition 3.22 and Theorem 3.5 As we have proved in general, the map d_2 takes into account the reductions applied to the ambiguity. □

5.2 Regularity properties

We begin this section by recalling the definitions of d-*Calabi-Yau* and *twisted* d-*Calabi-Yau* algebras.

Let $d \in \mathbb{N}$. An associative algebra A is said to be d-*Calabi-Yau* algebra if it has finite global dimension, it has a resolution by finitely generated projective A-bimodules and there is an isomorphism f of A-bimodules

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A & \text{if } i = d. \end{cases} \tag{5.1}$$

The A-bimodule outer structure of A^e is used when computing $\operatorname{Ext}^i_{A^e}(A, A^e)$, while the isomorphism f takes account of the inner bimodule structure of A^e. Bocklandt proved in [11] that graded Calabi-Yau algebras come from a potential and Van den Bergh [35] generalized this result to complete algebras with respect to the I-adic topology.

The definition of *twisted* d-*Calabi-Yau algebras* is very similar. Recall that if σ is an algebra automorphism of A, it is common to denote by A_σ the A-bimodule with A as underlying vector space and action of $A \otimes_k A^{op}$ given by: $(a \otimes b) \cdot x = ax\sigma(b)$. That is, the action is twisted on the right by the automorphism σ.

An algebra A is said to be *twisted* d-*Calabi-Yau* if it has finite global dimension, it has a resolution by finitely generated projective A-bimodules and there is an isomorphism f of A-bimodules

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A_\sigma & \text{if } i = d. \end{cases} \tag{5.2}$$

for some algebra automorphism σ of A.

Now we return to down-up algebras. So let k be a field and $\alpha, \beta, \gamma \in k$ and denote $A = A(\alpha, \beta, \gamma)$. Also, denote C the free A-bimodule resolution of A given in Section 5.1.

Remark 5.2. Before turning to the main result of this section, we describe an A-bimodule isomorphisms that we use. Let V be a k-vector space of finite dimension. Consider the k-linear morphism

$$\begin{aligned} \Phi : A \otimes_k V^* \otimes_k A &\longrightarrow \operatorname{Hom}_{A^e}(A \otimes_k V \otimes_k A, A^e) \\ a \otimes \varphi \otimes b &\mapsto [1 \otimes v \otimes 1 \mapsto \varphi(v) b \otimes a] \end{aligned}$$

Let us check that Φ is a morphism of A^e-right-modules when we consider the outer A-bimodule structure on $A \otimes_k V^* \otimes_k A$ and the A-bimodule structure on $\operatorname{Hom}_{A^e}(A \otimes_k V \otimes_k A, A^e)$ induced by the inner action on A^e, both of them seen as A^e-right-modules structures. It is enough to check it for elementary tensors, so let $v \in V$ and $a \otimes \varphi \otimes b \in A \otimes_k V^* \otimes_k A$ and $c \otimes d \in A^e$,

$$\begin{aligned} \Phi((a \otimes \varphi \otimes b) \cdot (c \otimes d))(v) &= \Phi(da \otimes \varphi \otimes bc)(v) = \varphi(v) bc \otimes da, \\ (\Phi(a \otimes \varphi \otimes b) \cdot (a \otimes b))(v) &= (\Phi(a \otimes \varphi \otimes b)(v)) \cdot (a \otimes b) \\ &= (\varphi(v) b \otimes a) \cdot (c \otimes d) \\ &= \varphi(v) bc \otimes da. \end{aligned}$$

The inverse of Φ is, fixing a k-basis $\{v_1, \dots, v_n\}$ of V and its dual basis $\{\varphi_1, \dots, \varphi_n\}$ of V^*,

$$\Phi^{-1} : \mathrm{Hom}_{A^e}(A \otimes_k V \otimes_k A, A^e) \longrightarrow A \otimes_k V^* \otimes_k A$$
$$f \mapsto \sum_{i,j} b^i_j \otimes \varphi_i \otimes a^i_j,$$

where $f(v_i) = \sum_j a^i_j \otimes b^i_j$.

After applying these identifications to $\mathrm{Hom}_{A^e}(C, A^e)$, we obtain the following complex of k-vector spaces

$$0 \longrightarrow A \otimes A \xrightarrow{\delta_0^*} A \otimes (kD \oplus kU) \otimes A \xrightarrow{d_1^*} A \otimes (kD^2U \oplus kDU^2) \otimes A \xrightarrow{d_2^*} A \otimes kD^2U^2 \otimes A \longrightarrow 0,$$

where $\otimes$ denotes $\otimes_k$, $\{D, U\}$ denotes the dual basis of $\{d, u\}$ and, accordingly, we denote with capital letters the dual bases of the other spaces.

The maps in the complex are, explicitely:

$$\begin{aligned}
\delta_0^*(1 \otimes 1) &= 1 \otimes D \otimes d - d \otimes D \otimes 1 + 1 \otimes U \otimes u - u \otimes U \otimes 1 \\
d_1^*(1 \otimes U \otimes 1) &= 1 \otimes D^2U \otimes d^2 - \alpha d \otimes D^2U \otimes d - \beta d^2 \otimes D^2U \otimes 1 + u \otimes DU^2 \otimes d \\
&\quad + 1 \otimes DU^2 \otimes du - \alpha du \otimes DU^2 \otimes 1 - \alpha \otimes DU^2 \otimes ud - \beta ud \otimes DU^2 \otimes 1 \\
&\quad - \beta d \otimes DU^2 \otimes u - \gamma \otimes DU^2 \otimes 1. \\
d_1^*(1 \otimes D \otimes 1) &= du \otimes D^2U \otimes 1 + u \otimes D^2U \otimes d - \alpha ud \otimes D^2U \otimes 1 - \alpha \otimes D^2U \otimes du \\
&\quad - \beta d \otimes D^2U \otimes u - \beta \otimes D^2U \otimes ud - \gamma \otimes D^2U \otimes 1 + u^2 \otimes DU^2 \otimes 1 \\
&\quad - \alpha u \otimes DU^2 \otimes u - \beta \otimes DU^2 \otimes u^2. \\
d_2^*(1 \otimes DU^2 \otimes 1) &= 1 \otimes D^2U^2 \otimes d + \beta d \otimes D^2U^2 \otimes 1, \\
d_2^*(1 \otimes D^2U \otimes 1) &= -u \otimes D^2U^2 \otimes 1 - \beta \otimes D^2U^2 \otimes u.
\end{aligned}$$

Notice that the homology of this complex is isomorphic to $\mathrm{Ext}^{\bullet}_{A^e}(A, A^e)$.

Definition 5.3. For every $\lambda \in k^\times$, define σ_λ to be the algebra automorphism $\sigma_\lambda : A \longrightarrow A$ given by $\sigma_\lambda(d) = \lambda d$ and $\sigma_\lambda(u) = \lambda^{-1}u$.

Lemma 5.4. *Suppose $\beta \neq 0$ and let C be the projective resolution of A as A-bimodule. There is an isomorphism of A-bimodule-complexes* $\mathrm{Hom}_{A^e}(C, A^e) \cong A_\sigma \otimes_A C$.

Proof. Consider the following isomorphisms of A-bimodules

$$
\begin{aligned}
&\psi_0 : A \otimes_E A \longrightarrow A \otimes_E kd^2u^2 \otimes_E A,\\
&\qquad \psi_0(1 \otimes 1) = 1 \otimes d^2u^2 \otimes 1,\\
&\psi_1 : A \otimes_E (kD \oplus kU) \otimes_E A \longrightarrow A \otimes_E (kd^2u \oplus kdu^2) \otimes_E A\\
&\qquad \psi_1(1 \otimes D \otimes 1) = \beta \otimes du^2 \otimes 1, \text{ and } \psi_1(1 \otimes U \otimes 1) = -1 \otimes d^2u \otimes 1\\
&\psi_2 : A \otimes_E (kD^2U \oplus kDU^2) \otimes_E A \longrightarrow A \otimes_E (kd \oplus ku) \otimes_E A,\\
&\qquad \psi_2(1 \otimes D^2U \otimes 1) = \beta \otimes u \otimes 1, \text{ and } \psi_2(1 \otimes DU^2 \otimes 1) = -1 \otimes d \otimes 1\\
&\psi_3 : A \otimes_E kD^2U^2 \otimes_E \longrightarrow A \otimes_E A\\
&\qquad \psi_3(1 \otimes D^2U^2 \otimes 1) = 1 \otimes 1.
\end{aligned}
$$

The following diagram commutes

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A \otimes A & \xrightarrow{\delta_0^*} & A \otimes (k\mathcal{A}_0)^* \otimes A & \xrightarrow{d_1^*} & A \otimes (k\mathcal{A}_1)^* \otimes A & \xrightarrow{d_2^*} & A \otimes (kA_2)^* \otimes A & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\
0 & \longrightarrow & A \otimes k\mathcal{A}_2 \otimes A & \xrightarrow{\overline{d}_0} & A \otimes k\mathcal{A}_1 \otimes A & \xrightarrow{\overline{d}_1} & A \otimes k\mathcal{A}_0 \otimes A & \xrightarrow{\overline{d}_2} & A \otimes A & \longrightarrow & 0
\end{array}
$$

where $\otimes$ denotes $\otimes_k$ and $\overline{d}_0$ is given by

$$\overline{d}_0(1 \otimes d^2u^2 \otimes 1) = \beta \otimes du^2 \otimes d - \beta d \otimes du^2 \otimes 1 - 1 \otimes d^2u \otimes u + u \otimes d^2u \otimes 1.$$

$\overline{d}_1$ is

$$
\begin{aligned}
\overline{d}_1(1 \otimes d^2u \otimes 1) &= 1 \otimes d \otimes du - \beta d \otimes d \otimes u + \beta^2 d^2 \otimes u \otimes 1\\
&\quad - \alpha(1 \otimes d \otimes ud - \beta d \otimes u \otimes d + du \otimes d \otimes 1)\\
&\quad - \beta(1 \otimes u \otimes d^2 - \beta^{-1} u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1\\
\overline{d}_1(1 \otimes du^2 \otimes 1) &= 1 \otimes d \otimes u^2 - \beta d \otimes u \otimes u + du \otimes u \otimes 1\\
&\quad - \alpha(1 \otimes u \otimes du - \beta^{-1} u \otimes d \otimes u + ud \otimes u \otimes 1)\\
&\quad - \beta(1 \otimes u \otimes ud - \beta^{-1} u \otimes u \otimes d - \beta^{-2} u^2 \otimes d \otimes 1) - \gamma \otimes u \otimes 1
\end{aligned}
$$

and $\overline{d}_2$ is

$$\overline{d}_2(1 \otimes u \otimes 1) = -\beta^{-1} u \otimes 1 - 1 \otimes u, \qquad \overline{d}_2(1 \otimes d \otimes 1) = -\beta d \otimes 1 - 1 \otimes d,$$

On the other hand, if V is a k-vector space, then

$$
\begin{aligned}
&A \otimes_k V \otimes_k A \longrightarrow A_\sigma \otimes_A (A \otimes_k V \otimes_k A)\\
&\qquad a \otimes v \otimes b \mapsto 1 \otimes (\sigma^{-1}(a) \otimes v \otimes b)
\end{aligned}
$$

is an isomorphism of A-bimodules. Under this identification, the above complex becomes $A_\sigma \otimes_A C$. □

Next we state and prove the main result of this section.

Proposition 5.5. *The algebra* $A(\alpha,\beta,\gamma)$ *is twisted 3-Calabi-Yau if and only if* $\beta \neq 0$*, in which case* $HH^3(A, A^e) \cong A_\sigma$*, for* $\sigma = \sigma_{-\beta}$ *given in Defintion 5.3. Moreover,* $A(\alpha,\beta,\gamma)$ *is 3-Calabi-Yau if and only if* $\beta = -1$.

Proof. Suppose $\beta = 0$. By Remark 5.2 we deduce that

$$\mathrm{Ext}^3_{A^e}(A, A^e) \cong \frac{A \otimes_k kD^2U^2 \otimes_k A}{\mathrm{Im}(d_2^*)} \cong \frac{A \otimes_k A}{\langle 1 \otimes d, u \otimes 1\rangle},$$

and so the endomorphism $u \cdot (-) : \mathrm{Ext}^3_{A^e}(A, A^e) \longrightarrow \mathrm{Ext}^3_{A^e}(A, A^e)$ given by the action on the left by u is not injective. On the other hand, since the set $\{u^i(du)^j d^k : i, j, k \in \mathbb{N}_0\}$ is a basis of A as k-vector space, we obtain that $u \cdot (-) : A_\sigma \longrightarrow A_\sigma$ is injective for every automorphism σ. Therefore there is no automorphism σ of A such that $\mathrm{Ext}^3_{A^3}(A, A^e)$ is isomorphic to A_σ as A-bimodules, and in particular A is not twisted 3-Calabi-Yau.

Let $\beta \neq 0$. From Lemma 5.4 and the fact that A_σ is A-projective, we obtain that

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq 3, \\ A_\sigma & \text{if } i = 3. \end{cases}$$

where $\sigma = \sigma_{-\beta}$, and therefore A is twisted 3-Calabi-Yau. If $\beta = -1$, then $\sigma = \mathrm{id}$ and so A is 3-Calabi-Yau. Conversely, if A is 3-Calabi-Yau, then $\beta \neq 0$ and $A_\sigma \cong A$. Suppose $f : A \longrightarrow A_\sigma$ is an isomorphism of A-bimodules. Observe that $1 = f(f^{-1}(1)) = f^{-1}(1)f(1) = f(1)\sigma(f^{-1}(1))$. That is, $f(1)$ is inversible on the left and on the right, and therefore it is a unit. By Lemma 1.3 in [16], any unit belongs to $k^\times$. Then,

$$df(1) = f(d) = f(1)\sigma(d) = -\beta f(1)d = -\beta df(1).$$

Recall that A is a domain since $\beta \neq 0$. From the above equation we deduce $\beta = -1$. □

5.3 Monomial down-up algebras

Recall that an algebra is said to be monomial if it is isomorphic to kQ/I, where Q is a quiver with a finite set of vertices and I is a two-sided ideal generated by paths of length at least 2. The goal of this section is to prove the following proposition.

Proposition 5.6. *The down-up algebra* $A(\alpha,\beta,\gamma)$ *is monomial if and only if* $\alpha = \beta = \gamma = 0$.

Notice that the algebra $A(0, 0, 0)$ is monomial by definition, so we only have to prove the *only if* part of the proposition. Before giving the proof, we state and prove a series of lemmas.

Definition 5.7. Let A be a k-algebra. Denote J_A the two sided ideal in A generated by the set $X_A := \{xy - yx : x, y \in A\}$. Define $\hat{A} := A/J_A$.

Lemma 5.8. *Let* A *and* B *be* k*-algebras. If* $A \cong B$*, then* $\hat{A} \cong \hat{B}$*. Both isomorphisms being of* k*-algebras.*

Proof. The isomorphism $\varphi : A \longrightarrow B$ verifies $\varphi(X_A) = X_B$ and so $\varphi(J_A) = J_B$. □

In order to state the next lemma we need some definitions. Let Q be a quiver with a finite set of vertices. For $e, e' \in Q_0$, define $eQ_1e' := \{\alpha \in Q_1 : t(\alpha) = e, s(\alpha) = e'\}$, where t and s are the usual target and source maps. Also, consider the k-algebra $B_e := k[X_\alpha : \alpha \in eQ_1e]$. In other words, B_e is the polynomial algebra having $\#eQ_1e$ variables indexed by the elements of this set. In case $eQ_1e = \emptyset$, we set $B_e := k$. If I is a two-sided ideal in kQ generated by paths of length at least 2, define I_e to be the ideal in B_e generated by the set $\{X_{\alpha_n} \cdots X_{\alpha_1} : \alpha_n \cdots \alpha_1 \in I, \alpha_i \in eQ_1e\}$.

Lemma 5.9. *Let* Q *be a quiver with a finite set of vertices and* I *a two-sided ideal in* kQ *generated by paths of length at least 2. Denote* $Q_0 = \{e_0, \dots, e_m\}$ *and* $B := kQ/I$*. There is an isomorphism of* k*-algebras* $\hat{B} \cong B_{e_1}/I_{e_1} \oplus \cdots \oplus B_{e_n}/I_{e_n}$

Proof. For $x \in B$ and $f \in B_e$, denote $\overline{x}$ the class of x in $\hat{B}$ and $\overline{f}$ the class of f in B_e/I_e. The map $\varphi : B_{e_1} \oplus \cdots \oplus B_{e_n} \longrightarrow \hat{B}$ sending each X_α to $\overline{\alpha}$ descends to the quotient $\overline{\varphi} : B_{e_1}/I_{e_1} \oplus \cdots \oplus B_{e_n}/I_{e_n}$. On the other hand, the algebra kQ is the free k-algebra on the set $Q_0 \cup Q_1$ modulo the two-sided ideal generated by the set

$$R := \{e_i e_j : 1 \leq i, j \leq n, i \neq j\} \cup \{t(\alpha)\alpha - \alpha : \alpha \in Q_1\} \cup \{\alpha s(\alpha) - \alpha : \alpha \in Q_1\}.$$

The set map $\psi : R \longrightarrow B_{e_1}/I_{e_1} \oplus \cdots \oplus B_{e_n}/I_{e_n}$ defined by

$$\begin{cases} \psi(e_i) = 1_i & \text{, where } 1_i \text{ is the unit in the } i\text{-th component, for } 1 \leq i \leq n, \\ \psi(\alpha) = \overline{X_\alpha} & \text{in the component } B_e/I_e \text{, if } \alpha \in eQ_1e, \\ \psi(\alpha) = 0 & \text{, if } \alpha \text{ does not belong to } eQ_1e \text{ for any } e \in Q_0 \end{cases}$$

induces a k-algebra morphism $\psi : kQ \longrightarrow B_{e_1}/I_{e_1} \oplus \cdots \oplus B_{e_n}/I_{e_n}$ that sends R to zero, and so it descends to B. Since $B_{e_1}/I_{e_1} \oplus \cdots \oplus B_{e_n}/I_{e_n}$ is commutative, it also descends to $\hat{B}$. Denote this map by $\overline{\psi}$. The maps $\overline{\psi}$ and $\overline{\varphi}$ are inverse of each other, giving the desired isomorphism. □

Lemma 5.10. *Let* $B = kQ/I$ *be a monomial algebra. For each* $e \in Q_0$ *denote by* T_e *the simple* B*-module corresponding to the vertex* e*. If* $e, e' \in Q_0$*, then* $\#eQ_1e' = \dim_k(\mathrm{Tor}_1^A(T_e, T_{e'}))$*. Moreover, if* Q *has only one vertex* e*, then*

$$\dim_k(\mathrm{Tor}_1^A(T_e, T_e)) = \sup\{\dim_k(\mathrm{Tor}_1^B(T_1, T_2) : T_1, T_2 \textit{ are one-dimensional } B\textit{-modules}\}.$$

Proof. Let $e, e' \in Q_0$. Bardzell's resolution of B starts as

$$\cdots \to B \otimes_E kQ_1 \otimes_E B \to B \otimes_E B \to B \to 0.$$

Apply the functor $T_e \otimes_B (-) \otimes_B T_{e'}$ and obtain the following complex

$$\cdots \to T_e \otimes_k keQ_1e' \otimes_k T_{e'} \to T_e \otimes_k T_{e'} \to 0,$$

whose homology is isomorphic to $\mathrm{Tor}^A_\bullet(T_e, T_{e'})$. Since Bardzell's resolution is minimal and every arrow acts as zero on T_e and $T_{e'}$, we obtain that the differentials of this complex are null. Therefore, $\mathrm{Tor}^A_1(T_e, T_{e'}) \cong T_e \otimes_k keQ_1e' \otimes_k T_{e'} \cong kQ_1$, from where we deduce $\dim_k(\mathrm{Tor}^A_1(T_e, T_{e'})) = \#eQ_1e'$.

As for the second assertion, the same argument shows that if T_1, T_2 are one-dimensional B-modules, then the homology of the complex

$$\cdots \to T_1 \otimes_k kQ_1 \otimes_k T_2 \to T_1 \otimes_k T_2 \to 0,$$

is isomorphic to $\mathrm{Tor}^A_\bullet(T_1, T_2)$. It follows that $\dim_k(\mathrm{Tor}^A_1(T_1, T_2)) \leq \dim_k(kQ_1) = \#Q_1$. From the first part of the lemma we obtain $\#Q_1 = \dim_k(\mathrm{Tor}^A_1(T_e, T_e))$, where e is the vertex of Q. □

Lemma 5.11. *Let $A = A(\alpha, 0, \gamma)$ and T_1, T_2 be one-dimensional A-modules.*

- *If $\gamma = 0$, then $\dim_k(\mathrm{Tor}^A_1(T_1, T_2)) \leq 2$ and $\dim_k(\mathrm{Tor}^A_1(k, k)) = 2$. Moreover, if $\alpha \neq 1$ and $T_1 \neq k$, then $\dim_k(\mathrm{Tor}^A_1(T_1, T_1)) = 1$.*
- *If $\gamma \neq 0$ and $\alpha = 1$, then $\dim_k(\mathrm{Tor}^A_1(T_1, T_2)) = 0$.*
- *If $\gamma \neq 0$ and $\alpha \neq 1$, then $\dim_k(\mathrm{Tor}^A_1(T_1, T_2) \leq 1$.*

Proof. Let T_1, T_2 be one-dimensional A-modules with bases $\{v_1\}$ and $\{v_2\}$, respectively. Let $\delta_1, \delta_2, \mu_1, \mu_2 \in k$ such that $d \cdot v_1 = \delta_i v_i$ and $u \cdot v_i = \mu_i v_i$ for $i = 1, 2$. In A we have the relations $d^2u - \alpha dud - \gamma d = 0 = du^2 - \alpha udu - \gamma u$, and so, for $i = 1, 2$,

$$\begin{aligned} &\delta_i((1-\alpha)\delta_i\mu_i - \gamma) = 0, \\ &\mu_i((1-\alpha)\delta_i\mu_i - \gamma) = 0. \end{aligned} \tag{5.3}$$

Applying the functor $T_1 \otimes_k (-) \otimes_k T_2$ to the resolution of A given in Section 5.1 we obtain the following complex k-vector spaces whose homology is isomorphic to $\mathrm{Tor}^A_\bullet(T_1, T_2)$,

$$0 \to kd^2u^2 \xrightarrow{f_2} kd^2u \oplus kdu^2 \xrightarrow{f_1} kd \oplus ku \xrightarrow{f_0} k \to 0,$$

where

$$\begin{aligned} f_0(d) &= \delta_2 - \delta_1, \\ f_0(u) &= \mu_2 - \mu_1, \\ f_1(d^2u) &= ((1-\alpha)\delta_1\mu_1 + \delta_2(\mu_1 - \alpha\mu_2) - \gamma)d + \delta_2(\delta_2 - \alpha\delta_1)u. \\ f_1(du^2) &= \mu_1(\mu_1 - \alpha\mu_2)d + ((1-\alpha)\delta_2\mu_2 + \mu_1(\delta_2 - \alpha\delta_1) - \gamma)u. \end{aligned}$$

The fact that the space $kd \oplus ku$ is two-dimensional implies $\dim_k(\mathrm{Tor}_1^A(T_1, T_2)) \leq 2$ for all $\alpha, \gamma \in k$.

- If $\gamma = 0$ and $T_1 = T_2 = k$, then $f_0 = 0$,$f_1 = 0$ and therefore $\dim_k(\mathrm{Tor}_1^A(k, k)) = 2$. If $\alpha \neq 1$ and $T_1 = T_2 \neq k$, we see from equations 5.3 that $\delta_1 = 0$ or $\mu_1 = 0$, but not both. Then $f_0 = 0$ and f_1 is of rank 1, and we obtain $\dim_k(\mathrm{Tor}_1^A(T_1, T_1)) = 1$.

- If $\gamma \neq 0$ and $\alpha = 1$, then from equations 5.3 we deduce $\delta_1 = \delta_2 = \mu_1 = \mu_2 = 0$. Then, $f_0 = 0$, $f_1(d^2u) = -\gamma d$, $f_1(du^2) = -\gamma u$ and so $\mathrm{Tor}_1^A(T_1, T_2) = 0$.

- If $\gamma \neq 0$ and $\alpha \neq 1$ there are several cases:

  – Case $(\delta_1, \mu_1) = (\delta_2, \mu_2) = (0, 0)$. In this case $f_0 = 0$, $f_1(d) = -\gamma d$, $f_1(u) = -\gamma u$. Then $\mathrm{Tor}_1^A(T_1, T_2) = 0$.

  – Case $(\delta_1, \mu_1) = (0, 0), (\delta_2, \mu_2) \neq (0, 0)$. From equations 5.3 we see that $\delta_2 = 0$ if and only if $\mu_2 = 0$. Then $\delta_2 \neq 0$ and $\mu_2 \neq 0$. From this we see that $f_0 \neq 0$ and $\dim_k(\mathrm{Tor}_1^A(T_1, T_2)) \leq 1$.

  – Case $(\delta_1, \mu_1) \neq (0, 0)$, $(\delta_2, \mu_2) \neq (0, 0)$ and $(\delta_1, \mu_1) \neq (\delta_2, \mu_2)$. In this case $f_0 \neq 0$ and so $\dim_k(\mathrm{Tor}_1^A(T_1, T_2)) \leq 1$.

  – Case $(\delta_1, \mu_1) = (\delta_2, \mu_2) \neq (0, 0)$. Once again, from equations 5.3 we deduce that $\delta_2 = 0$ if and only if $\mu_2 = 0$, and therefore δ_1 and μ_1 are not zero. Also, we obtain $(1 - \alpha)\delta_1\mu_1 - \gamma = 0$. Then $f_0 = 0$ and

  $$f_1(d^2u) = \delta_1(1 - \alpha)(\mu_1 d + \delta_1 u),$$
  $$f_1(du^2) = \mu_1(1 - \alpha)(\mu_1 d + \delta_1 u).$$

  The fact that $\alpha \neq 1$ implies $f_1(d^2u) \neq 0$ and $f_1(du^2) \neq 0$. Moreover, $\mu_1 f_1(d^2u) = \delta_1 f_1(du^2)$. We deduce $\dim_k(\mathrm{Tor}_1^A(T_1, T_2)) = 1$.

□

We now proceed to prove Proposition 5.6. Let $A = A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in k$ with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let $B = kQ/I$ be a monomial algebra and suppose there exists an isomorphism of k-algebras $\varphi : A \longrightarrow B$. Since every down-up algebra is of global dimension 3, we deduce $I \neq$. This implies that B is not a domain and that $\beta \neq 0$, since any down-up algebra $A(\alpha', \beta', \gamma')$ with $\beta' \neq 0$ is a domain.

Suppose $\gamma = 0$. By Lemma 5.11,

$$\sup \dim_k(\mathrm{Tor}_1^A(T_1, T_2) : T_1, T_2 \text{ are one-dimensional } A\text{-modules}\} = 2.$$

Notice that $\hat{A} = k[d, u]/\langle(1 - \alpha)d^2u, (1 - \alpha)du^2\rangle$ is connected. Since $\hat{A} \cong \hat{B}$ we obtain that B is connected. Recall $B = kQ/I$ is a monomial algebra. By lemmas 5.9 and 5.10, Q has exactly one vertex and two arrows. Denote $Q_1 = \{x, y\}$. We have the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & kQ/I \\ {\scriptstyle\pi}\big\downarrow & \swarrow{\scriptstyle \pi\circ\varphi^{-1}} & \\ k_\alpha[d,u] & & \end{array}$$

where $k_\alpha[d,u] := A/\langle du - \alpha ud\rangle$. This algebra is called *the quantum plane* and it is a domain for all $\alpha \neq 0$. Let $p(x,y)$ be a path in I. The diagram shows that $p(\pi \circ \varphi^{-1}(x), \pi \circ \varphi^{-1}(y)) = 0$. Since $\gamma = 0$, we have $\alpha \neq 0$ and therefore $k_\alpha[d,u]$ is a domain. We deduce $\pi \circ \varphi^{-1}(x) = 0$ or $\pi \circ \varphi^{-1}(y) = 0$ and this implies that $k_\alpha[d,u]$ is generated by one element, which is a contradiction.

Now suppose $\gamma \neq 0$. If $\alpha = 1$, then Lemma 5.11 says that $\mathrm{Tor}_1^A(T_1, T_2) = 0$ for every pair of one-dimensional A-modules. Since $A \cong B$, this is also true for B and one-dimensional B-modules. By Lemma 5.10 Q has no arrows. This is impossible, and so $\alpha \neq 1$.

We have

$$\hat{A} = \frac{k[d,u]}{\langle d((1-\alpha)du - \gamma), u((1-\alpha)du-\gamma)\rangle} \cong k \oplus \frac{k[d,u]}{\langle (1-\alpha)du - \gamma\rangle}.$$

The last isomorphism comes from the fact that the ideals $\langle d,u\rangle$ and $\langle (1-\alpha)du - \gamma)\rangle$ in $k[d,u]$ are coprime. We will identify $\hat{A}$ with $k \oplus k[d,u]/\langle (1-\alpha)du - \gamma\rangle$. Lemma 5.11 shows that $\dim_k(\mathrm{Tor}_1^A(T_1,T_2)) \leq 1$ for all one-dimensional A-modules T_1, T_2. Therefore this is also verified by B and one-dimensional B-modules. By 5.10 there is at most one arrow between each pair of vertices in Q. In particular, there is at most one element in eQ_1e for every vertex e. Define $V = \{e \in Q_0 : \#eQ_1e = 1\}$ and for every $e \in V$, denote x_e the unique element in eQ_1e. The fact that A is of global dimension 3 implies that B is also of global dimension 3. In particular, Bardzell's resolution of B is of finite length. From this we obtain that $x_e^n \notin I$ for all $n \in \mathbb{N}$, otherwise Bardzell's resolution of B would be of infinite length. Therefore, using the notation of Lemma 5.9, this implies $B_e = k[X]$ and $I_e = 0$ for all $e \in V$, and $B_e = k$ for all $e \notin V$. By Lemma 5.9, we have

$$\hat{B} \cong \bigoplus_{e \notin V} k \oplus \bigoplus_{e \in V} k[X].$$

We identify $\hat{B}$ with the algebra on the right. Denote $\hat{\varphi} : \hat{A} \longrightarrow \hat{B}$ the isomorphism induced by $\varphi : A \longrightarrow B$. Consider the element $(1,u) \in \hat{A}$. We have

$$(1, (1-\alpha)\gamma^{-1}d) \cdot (1,u) = (1,1) \text{ in } \hat{A}.$$

This implies that $\hat{\varphi}((1,u))$ is a unit in $\hat{B}$. The units in $\hat{B}$ are contained in $k^n \subseteq \hat{B}$, where $n = \#Q_0$. The dimension of the vector space generated by the set $\{(1,u^i)\}_{i\in\mathbb{N}}$ in $\hat{A}$ is infinite dimensional, and the their images via $\hat{\varphi}$ generate a finite dimensional space since it is contained in k^n. This is a contradiction and we conclude the proof of Proposition 5.6.

5.4 The isomorphism problem

The isomorphism problem for down-up algebras was posed in [12] where the authors divide down-up algebras $A(\alpha, \beta, \gamma)$ into four types and show, by studying one-dimensional modules, that algebras of different types are not isomorphic. The division into types is the following,

(a) $\gamma = 0$, $\alpha + \beta = 1$,

(b) $\gamma = 0$, $\alpha + \beta \neq 1$,

(c) $\gamma \neq 0$, $\alpha + \beta = 1$,

(d) $\gamma \neq 0$, $\alpha + \beta \neq 1$,

As a consequence, the question of whether two down-up algebras are isomorphic can be restricted to each of the four types. In [16] the authors solve the isomorphism problem for noetherian down-up algebras of type (a), (b) and (c) for every field k, and also for noetherian algebras of type (d) when $\mathsf{char}(k) = 0$. Their solution focuses mainly on the commutative quotients of down-up algebras.

The purpose of this section is to provide the solution to the isomorphism problem for nonnoetherian down-up algebras for every field k. Namely, we obtain the following proposition.

Proposition 5.12. *Let* $\alpha, \gamma \in k$ *and* $A = A(\alpha, 0, \gamma)$ *and* $A' = A(\alpha', 0, \gamma')$ *be down-up algebras. The algebra* A *is isomorphic to* A' *if and only if* $\alpha = \alpha'$ *and* $\gamma = \lambda\gamma'$ *for some* $\lambda \in k^{\times}$.

We start by organizing the problem. Notice that the condition $\gamma = \lambda\gamma'$ for $\lambda \in k^{\times}$ is equivalent to the condition of γ and γ' being both zero or both nonzero. On the other hand, if $\gamma \neq 0$, then $A(\alpha, 0, \gamma)$ is isomorphic to $A(\alpha, 0, 1)$. This is done by reescaling d by γd. Also, observe that $A(\alpha, 0, 0)$ is not isomorphic to $A(\alpha', 0, 1)$ for all $\alpha, \alpha' \in k$, since they belong to different types. Putting this all together, we obtain that Proposition 5.12 is equivalent to the following two assertions:

If $A(\alpha, 0, 0)$ is isomorphic to $A(\alpha', 0, 0)$, then $\alpha = \alpha'$,
If $A(\alpha, 0, 1)$ is isomorphic to $A(\alpha', 0, 1)$, then $\alpha = \alpha'$.

Is this form of the proposition that we will prove. The first assertion is Proposition 5.15 and the second assertion is Proposition 5.18

Our approach is to recover information from very well studied noncommutative algebras that appear as quotients of down-up algebras only in the nonnoetherian cases $A(\alpha, 0, \gamma)$ for $\alpha \neq 0$. These noncommutative quotients are respectively, depending whether γ is equal to 0 or not, the quantum plane $k_\alpha[x, y]$ and the quantum Weyl algebra A^1_α. We recall their definition,

$$k_\alpha[x, y] := k\langle x, y : yx - \alpha xy = 0\rangle, \qquad A^1_\alpha := k\langle x, y : yx - \alpha xy = 1\rangle.$$

Lemma 5.13. *Let $\alpha, \gamma \in k$, $\alpha \neq 0$ and $A = A(\alpha, 0, \gamma)$ a down-up algebra. Denote $\omega := du - \alpha ud - \gamma$. If $\gamma = 0$, then $A/\langle\omega\rangle$ is isomorphic to $k_\alpha[x, y]$. If $\gamma \neq 0$, then $A/\langle\omega\rangle$ is isomorphic to A^1_α. Moreover, if $\gamma = 0$ or $\gamma = 1$, then the isomorphism sends the class of d to y and the class of u to x.*

Proof. The algebra $A(\alpha, 0, \gamma)$ is the algebra freely generated by letters d, u subject to the relations $d^2u - \alpha dud - \gamma d = 0$ and $du^2 - \alpha udu - \gamma u = 0$. Let Ω be the element $du - \alpha ud - \gamma$ in the free algebra. The projection of Ω in $A(\alpha, 0, \gamma)$ is ω. The relations defining A can be written as $d\Omega = 0$ and $\Omega u = 0$. Therefore, the algebra $A/\langle\omega\rangle$ is isomorphic to the algebra freely generated by letters d, u subject to the relation $\Omega = 0$. If $\gamma = 0$ this is exactly the definition of $k_\alpha[x, y]$. If $\gamma \neq 0$, then $\omega = \gamma^{-1}((\gamma d)u - \alpha u(\gamma d) - 1)$, and so $A/\langle\omega\rangle$ is the quantum Weyl algebra with $y = \gamma d$ and $x = u$. □

In [31] the authors find all isomorphisms and automorphisms for quantum Weyl algebras A^1_α in the case where $\alpha \in k^\times$ is not a root of unity. In [34], this result is generalized to the family of *quantum generalized Weyl algebras*, which is a family that contains the quantum plane and the quantum Weyl algebra for all values of $\alpha \in k^\times$. We recall some of their results in the cases relevant to us.

Theorem 5.14 ([31],[34])**.** *Let $\alpha, \alpha' \in k \setminus \{0, 1\}$.*

The two algebras $k_\alpha[x, y]$ and $k_{\alpha'}[x, y]$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. Moreover, if $\varphi : k_\alpha[x, y] \longrightarrow k_{\alpha^{-1}}[x, y]$ is an isomorphism and $\alpha \neq \alpha^{-1}$, then there exist $\lambda, \mu \in k^\times$ such that

$$\varphi(x) = \lambda y \text{ and } \varphi(y) = \mu x.$$

The two algebras A^1_α and $A^1_{\alpha'}$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. Also, if $\alpha \neq \alpha^{-1}$, every isomorphism $\eta : A^1_\alpha \longrightarrow A^1_{\alpha^{-1}}$ is of the form

$$\eta(x) = \lambda y \text{ and } \eta(y) = -\lambda^{-1}\alpha^{-1}x,$$

for some $\lambda \in k^\times$.

We are now in position to prove the first assertion.

Proposition 5.15. *Let $\alpha, \alpha' \in k$. If $A(\alpha, 0, 0)$ is isomorphic to $A(\alpha', 0, 0)$, then $\alpha = \alpha'$.*

Proof. Let $\alpha, \alpha' \in k$ and suppose $\varphi : A(\alpha, 0, 0) \longrightarrow A(\alpha', 0, 0)$ is an isomorphism. Denote $A := A(\alpha, 0, 0)$, $A' := A(\alpha', 0, 0)$ and write d' and u' the generators of A'.

If $\alpha = 1$, then A belongs to type (a), and so A' belongs to type (a) as well. This implies $\alpha' = 1$. If $\alpha = 0$, then A is monomial and therefore A' is monomial. By Proposition 5.6, we obtain $\alpha' = 0$.

Suppose $\alpha, \alpha' \in k \setminus \{0, 1\}$ and $\alpha \neq \alpha'$. Let $\omega := du - \alpha ud$ and $\omega' := d'u' - \alpha' u'd'$. By Lemma 5.13, we can identify $A/\langle\omega\rangle$ with $k_\alpha[x, y]$, where the canonical projection

$\pi : A \longrightarrow k_\alpha[x, y]$ sends d to y and u to x. We make the same identification with $A'/\langle\omega'\rangle$ and $k_{\alpha'}[x, y]$ and write π' the canonical projection. Define $\psi_1 := \pi' \circ \varphi : A \longrightarrow k_{\alpha'}[x, y]$. In A, we have the equalities $d\omega = 0$ and $\omega u = 0$. Therefore $\psi_1(d)\psi_1(\omega) = 0 = \psi_1(\omega)\psi_1(u)$. Observe $k_{\alpha'}[x, y]$ is a domain generated by $\psi(d)$ and $\psi_1(u)$. Thus, $\psi(d)$ and $\psi(u)$ are not zero and we obtain $\psi_1(\omega) = 0$. This implies that $\psi_1 = \overline{\psi}_1 \circ \pi$ with $\overline{\psi}_1 : k_\alpha[x, y] \longrightarrow k_{\alpha'}[x, y]$. In the other direction we obtain that $\psi_2 := \pi \circ \varphi^{-1}$ factors as $\psi_2 = \overline{\psi}_2 \circ \pi'$. Since $\overline{\psi}_1 \circ \overline{\psi}_2 \circ \pi' = \pi'$ and $\overline{\psi}_2 \circ \overline{\psi}_1 \circ \pi = \pi$, we deduce $\overline{\psi}_1$ is an isomorphism. The situation is illustrated by the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & A' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ k_\alpha[x, y] & \underset{\overline{\psi}_2}{\overset{\overline{\psi}_1}{\rightleftarrows}} & k_{\alpha'}[x, y] \end{array}$$

By Theorem 5.14, we obtain $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$. Since we are assuming $\alpha \neq \alpha'$, we deduce $\alpha' = \alpha^{-1}$. Theorem 5.14 implies that there exist $\lambda, \mu \in k^\times$ and $z_1, z_2 \in \langle\omega'\rangle$ such that $\varphi(u) = \lambda d' + z_1$ and $\varphi(d) = \mu u' + z_2$. Notice that A' is graded considering the generators d' and u' in weight 1. Under this grading, the elements z_1, z_2 are sums of homogeneous elements of degree at least 2. This follows from the fact that $z_1, z_2 \in \langle\omega'\rangle$ and $\omega' = du - \alpha ud$. On the other hand, $0 = d^2u - \alpha dud \in A$, and therefore $0 = \varphi(d)^2\varphi(u) - \alpha\varphi(d)\varphi(u)\varphi(d)$. The degree 2 component of this equality is $0 = (u')^2d' - \alpha u'd'u'$. Since the set $\{(u')^i(du)^j d^l : i, j, l \in \mathbb{N}_0\}$ is a k-basis of A', we arrive at a contradiction. This concludes with the proof of the proposition. □

We now turn our attention to the second assertion. Let $A = A(\alpha, 0, 1)$ for $\alpha \in k$. Recall that $\omega := du - \alpha ud - 1$. The set $\{u^i\omega^j d^l : i, j, l \geq 0\}$ is a k-basis of A. This is Lemma 2.2 in [36].

Lemma 5.16. *The set* $\{u^i\omega^j d^l : i, l \geq 0$ *and* $j \geq 1\}$ *is a* k*-linear basis of the two sided ideal* $\langle\omega\rangle$*, and, for each* $n \in \mathbb{N}$*, the set* $\{u^i\omega^j d^l : i, l \geq 0$ *and* $j \geq n\}$ *is a* k*-linear basis of* $\langle\omega\rangle^n$*.*

Proof. Let us first prove the first claim. Every element of the form $u^i\omega^j d^l$ with $j \geq 1$ belongs to $\langle\omega\rangle$, and so we only have to show that $\langle\omega\rangle$ is contained in the k-vector space with basis $\{u^i\omega^j d^l : i, l \geq 0$ and $j \geq 1\}$. Let $z \in \langle\omega\rangle$ and write $z = \sum_{i,j,l} \lambda_{i,j,l} u^i\omega^j d^l$ with $i, j, l \geq 0$ and $\lambda_{i,j,l} \in k$. By Lemma 5.13 we can identify $A/\langle\omega\rangle$ with A^1_α, and the canonical projection $\pi : A \longrightarrow A^1_\alpha$ sends u to x and d to y. The set $\{x^i y^l : i, l \geq 0\}$ is a basis of A^1_α. Therefore, $0 = \pi(z) = \sum_{i,l} \lambda_{i,0,l} x^i y^l$. From this we deduce $\lambda_{i,0,l} = 0$ for all $i, l \geq 0$.

Taking into account the description we now have for the elements of $\langle\omega\rangle$, we see that the elements of $\langle\omega\rangle^2$ are linear combinations of elements of the form $u^i\omega^j d^l u^{i'}\omega^{j'} d^{l'}$, with $j, j' \geq 1$. Similarly, the elements of $\langle\omega\rangle^n$ are linear combinations of n-fold products of the same type. Therefore, to prove the second claim, it is sufficient to show that for every $r, s \geq 0$, we have $\omega d^r u^s \omega = \sum_{i \geq 2} \lambda_i \omega^i$ for some $\lambda_i \in k$.

Let $r, s \geq 0$. Write $d^r u^s = \sum_{i,j,l\geq 0} \lambda_{i,j,l} u^i \omega^j d^l$. Recall that A is graded considering d in degree -1 and u in degree 1. Under this grading, the element ω is homogeneous of degree 0. We have

$$d^r u^s - \sum_{\substack{l-i=r-s \\ j\geq 0}} \lambda_{i,j,l} u^i \omega^j d^l = \sum_{l-i\neq r-s, j\geq 0} \lambda_{i,j,l} u^i \omega^j d^l.$$

The term on the left is homogeneous of degree $r-s$ and the term on the right is a sum of homogeneous elements neither of which is of degree $r-s$. Therefore both terms are equal to zero and we obtain

$$\omega d^r u^s \omega = \sum_{\substack{l-i=r-s \\ j\geq 0}} \lambda_{i,j,l} \omega u^i \omega^j d^l \omega.$$

Recall that the relations of the algebra A are $d\omega = 0$ and $\omega u = 0$. If $r - s \neq 0$, then $(i, l) \neq (0, 0)$ for all i, l with $l - i = r - s$. As a consequence, from the above equality we see that $\omega d^r u^s \omega = 0$. On the other hand, if $r = s$, then

$$\omega d^r u^r \omega = \sum_{j,l\geq 0} \lambda_{l,j,l} \omega u^l \omega^j d^l \omega = \sum_{j\geq 0} \lambda_{0,j,0} \omega^{j+2}.$$

□

Corollary 5.17. *The set* $\{[u^i \omega d^l] : i, l \geq 0\}$*, where* $[p]$ *denotes the class of an element* p *in* $\langle\omega\rangle / \langle\omega\rangle^2$*, is a* k*-linear basis of the* A*-bimodule* $\langle\omega\rangle / \langle\omega\rangle^2$*. On the other hand, if* $\alpha \neq 1$*, we have the following formulas,*

$$[u^i \omega d^l u] = \frac{\alpha^l - 1}{\alpha - 1} [u^i \omega d^{l-1}],$$
$$[d u^i \omega d^l] = \frac{\alpha^i - 1}{\alpha - 1} [u^{i-1} \omega d^l].$$

When $l = 0$ *or* $i = 0$*, the terms on the right are considered to be zero.*

Proof. The first claim follows directly from 5.16. To prove the first formula, we will fix $i \geq 0$ and proceed by induction on l. We have the equalities $\omega u = 0 = d\omega$, which proves the case $l = 0$. On the other hand, we have $\omega^2 = \omega(du - \alpha ud - 1) = \omega du - \omega$, that is $\omega du = \omega^2 + \omega$. Similarly $du\omega = \omega^2 + \omega$. Therefore, $[u^i \omega du] = [u^i \omega]$. Now, if $l \geq 2$, then

$$\begin{aligned}[u^i \omega d^l u] = [u^i \omega d^{l-2}(\alpha dud + d)] &= \alpha [u^i \omega d^{l-1} ud] + [u^i \omega d^{l-1}] \\ &= \alpha \frac{\alpha^{l-1} - 1}{\alpha - 1} [u^i \omega d^{l-2} d] + [u^i \omega d^{l-1}] \\ &= \frac{\alpha^l - 1}{\alpha - 1} [u^i \omega d^{l-1}].\end{aligned}$$

The second formula can be proved analogously. □

Proposition 5.18. *Let $\alpha, \alpha' \in k$. Then $A(\alpha, 0, 1)$ is isomorphic to $A(\alpha', 0, 1)$ if and only if $\alpha = \alpha'$.*

Proof. Let $\alpha, \alpha' \in k$. Denote $A := A(\alpha, 0, 1)$, $A' := A(\alpha', 0, 1)$ and denote d' and u' the generators of A'. Let $\varphi : A \longrightarrow A'$ be an isomorphism. Also, recall that $\omega = du - \alpha ud - 1$ and $\omega' = d'u' - \alpha' u'd' - 1$.

If $\alpha = 1$, then A belongs to type (a), and so does A'. Therefore $\alpha' = 1$. Now suppose $\alpha = 0$ and $\alpha' \neq 0$. By Lemma 5.13 we have that $A'/\langle\omega'\rangle$ can be identified with $A^1_{\alpha'}$ and if π' is the canonical projection, then $\pi'(u') = x$ and $\pi'(d') = y$. Let $\psi = \pi' \circ \varphi$. Since $d\omega = \omega u = 0$, we have $\psi(d)\psi(\omega) = \psi(\omega)\psi(u) = 0$. Notice that $\psi(d)$ and $\psi(u)$ generate $A^1_{\alpha'}$, and therefore they cannot be zero. We deduce $\psi(\omega) = 0$. Since $\omega = du - 1$, we obtain $\psi(d)\psi(u) = 1$. Also, $\psi(d)(\psi(u)\psi(d) - 1) = 0$ and we deduce $\psi(d)$ and $\psi(u)$ are units in $A^1_{\alpha'}$. If ε is a unit in $A^1_{\alpha'}$, then there is an automorphism $\eta : A^1_{\alpha'} \longrightarrow A^1_{\alpha'}$ defined by $\eta(x) = \epsilon x$ and $\eta(y) = \varepsilon^{-1} y$. Theorem 5.14 implies $\varepsilon \in k$. In our case this says that $\psi(d)$ and $\psi(u)$ belong to k, but this is a contradiction, since they generate $A^1_{\alpha'}$. The contradiction comes from the assumption $\alpha = 0$.

Suppose $\alpha, \alpha' \in k \setminus \{0, 1\}$ and $\alpha \neq \alpha'$. By the same arguments as before, we have that $\psi := \pi' \circ \varphi : A \longrightarrow A^1_{\alpha'}$ induces an isomorphism $\overline{\psi} : A^1_{\alpha} \longrightarrow A^1_{\alpha'}$. Theorem 5.14 implies $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$. Since we are asssuming $\alpha \neq \alpha'$, we deduce $\alpha' = \alpha^{-1}$. Again, by Theorem 5.14 we obtain that there exist $\lambda \in k^\times$ and $z_1, z_2 \in \langle\omega\rangle$ such that

$$\begin{aligned} \varphi^{-1}(d') &= -\lambda^{-1}\alpha u + z_1, \\ \varphi^{-1}(u') &= \lambda d + z_2. \end{aligned}$$

By reescaling the variables d, u, we can assume $\lambda = 1$. The equality $(d')^2 u' - \alpha^{-1} d'u'd' - d' = 0$ implies $\varphi(d')^2\varphi(u') - \alpha^{-1}\varphi(d')\varphi(u')\varphi(d') - \varphi(d') = 0$. Expand it and denote z the sum of all the terms in which at least two factors of z_1 or z_2 are involved. We find

$$\begin{aligned} 0 &= \alpha^2 u^2 d - \alpha udu + \alpha u + \alpha^2 u^2 z_2 - \alpha u z_1 d - \alpha z_1 ud + udz_1 - \alpha u z_2 u + z_1 du - z_1 + z \\ &= -\alpha u\omega + \alpha(\alpha u^2 z_2 - u z_1 d) + (udz_1 - \alpha u z_2 u) + z_1\omega + z \in \langle\omega\rangle. \end{aligned}$$

Notice that $z_1\omega, z \in \langle\omega\rangle^2$. Taking class modulo $\langle\omega\rangle^2$ we obtain

$$\alpha[u\omega] + \alpha([uz_1 d] - \alpha[u^2 z_2]) = [udz_1] - \alpha[uz_2 u].$$

Write $z_1 = \sum_{i,l\geq 0, j\geq 1} \lambda_{i,j,l} u^i \omega^j d^l$ and $z_2 = \sum_{i,l\geq 0, j\geq 1} \mu_{i,j,l} u^i \omega^j d^l$. Using the formulas of Corollary 5.17 we obtain

$$\begin{aligned} &\alpha[u\omega] + \sum_{i,l\geq 0} \alpha(\lambda_{i,1,l}[u^{i+1}\omega d^{l+1}] - \alpha\mu_{i,1,l}[u^{i+2}\omega d^l]) = \\ &= \sum_{i\geq 1, l\geq 0} \lambda_{i,1,l}\frac{\alpha^i - 1}{\alpha - 1}[u^i\omega d^l] - \sum_{i\geq 0, l\geq 1} \alpha\mu_{i,1,l}\frac{\alpha^l - 1}{\alpha - 1}[u^{i+1}\omega d^{l-1}]. \end{aligned}$$

By Corollary 5.17, the set $\{[u^i \omega d^l] : i, l \geq 0\}$ is a k-linear basis of $\langle\omega\rangle/\langle\omega\rangle^2$. For $m \geq 0$, define $\Lambda_m := \lambda_{m+1,1,m} - \alpha\mu_{m,1,m+1}$. Regarding, for each $m \geq 0$, the coefficient corresponding to the term $[u^{m+1}\omega d^m]$ in the last equation, we deduce

$$\alpha = \Lambda_0,$$
$$\alpha\Lambda_{m-1} = \frac{\alpha^{m+1} - 1}{\alpha - 1}\Lambda_m, \text{ for } m \geq 1.$$

Since $\alpha \neq 0$, we obtain $\Lambda_0 \neq 0$. Inductively, if $\Lambda_{m-1} \neq 0$, then the above equality implies $\Lambda_m \neq 0$. Therefore $\Lambda_m \neq 0$ for all $l \geq 0$. As a consequence, either the $\lambda_{m+1,1,m} \neq 0$ for infinitely many $m \in \mathbb{N}$, or $\mu_{m,1,m+1} \neq 0$ for infinitely many $m \in \mathbb{N}$. This is a contradiction that comes from the assumption $\alpha \neq \alpha'$. □

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